THEORY OF PLANE CURVES

PART II

CUBIC AND QUARTIC CURVES

LECTURES

ON THE

THEORY OF PLANE CURVES

DELIVERED TO POST-GRADUATE STUDENTS
IN THE UNIVERSITY OF CALCUTTA

BY

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PREFACE

The subject of Analytical Geometry covers so extensive a field that it is by no means easy to decide what to omit and what to insert ; a word is therefore necessary to explain the plan adopted in this book, which, I trust, will prove a useful introduction to the higher branches of the subject and will facilitate the study of a variety of algebraic curves.

In the preparation of these lectures, I have endeavoured to present the subject in clear and concise terms to the student commencing a systematic study of the properties of algebraic curves, especially of Cubics and Quartics. In the portion of the book devoted to the discussion of cubic curves, I have not confined myself exclusively to the application of analytical methods, but have availed myself of the methods of Geometry whenever simplicity could be gained thereby. One prominent feature of the present work is that properties of cubic curves have been exhaustively discussed with special reference to points of inflexion and harmonic polars ; in this connection, canonical forms have at times been found of great use, A separate chapter has been allotted to the discussion of some special cubic curves of historic importance, and their most general properties ; but no systematic analysis has been attempted, lest the young student should feel embarrassed ; only

general characteristics of these curves have been outlined which will supply sufficient material for independent thinking in more advanced stages.

The subject of quartic curves is too extensive to be adequately considered in a small work like this. I have therefore confined the discussion chiefly to the most prominent characteristics of these curves. One chapter has been devoted to the consideration of bicircular quartics with special reference to their mode of generation. In fact, this chapter, together with a note in Appendix I, is mainly based on the well-known Memoir on bicircular quartics by Dr. Casey, published in the Transactions of the Royal Irish Academy 1869. Circular cubics have been studied with much advantage, regarded as degenerate bicircular quartics. In the last chapter are considered some well known quartic curves, most of which are bicircular or are cartesians. A similar consideration, as in the case of cubic curves, has led me to restrict my discourse only to the general properties of these curves. I have intentionally avoided the discussion of Roulettes, Cycloids, etc., reserving the topics for a future occasion. The reader who desires to study the subject from a higher standpoint can conveniently consult the following works—Clebsch—Leçons sur la Géométrie ; Chasles—Histoire de la Géométrie. In Appendix II, a note on Trinodal Quartics has been inserted. This was communicated to me by Rai A. C. Bose, Bahadur, M.A., Controller of Examinations, University of Calcutta.

In studying singular points on cubics and quartics, I have retained the common phrase "non-singular" to designate a curve which has no double point or multiple point, although it has been pointed out by Prof. Basset that this is a misnomer ; for, he says, Plücker had shown that all algebraic curves except conics possess singularities and accordingly he introduced the term "anautotomic" in preference to the pharse " non-singular " commonly in use.

In concluding this preface, I desire to say that, in addition to the works of authors cited in the preface to the first part, I have consulted, with much advantage, some notes on Cubics and Quartics furnished by my colleague Dr. H. D. Bagchi, M.A., Ph.D., and that I am indebted for some valuable hints to Rai A. C. Bose, Bahadur, M.A. Once more I must acknowledge myself in the highest degree indebted to Sir Asutosh Mookerjee, Kt., President of the Council of Post-Graduate Teaching in Arts, for his extreme kindness in encouraging me to revise these lecture-notes for the press, and to the authorities of the University of Calcutta for publishing them. Finally, I must thank the Staff of the Calcutta University Press, but for whose untiring energy and ready co-operation, the second part of the book could not have seen the light of day before December next.

University of Calcutta,

August, 1919.

S. M. GANGULI.

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CHAPTER X.

Curves of the third order—Cubic Curves.

139. The general equation of a curve of the third order involves ten arbitrary constants. We may divide out the whole expression by any *one* of the constants, and hence the number of disposable constants in the equation of a cubic curve is *nine* and a cubic can be made to pass through any nine arbitrary points; or nine arbitrary points will determine a curve of the third order uniquely.

The general equation of a cubic curve in Cartesian co-ordinates can be written as—

$$ax^3+3bx^2y+3cxy^2+dy^3+a'x^2+2h'xy+b'y^2+lx+my+n=o. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Or, in any system of homogeneous co-ordinates, it may be taken as $ax^3+3bx^2y+3cxy^2+dy^3+a'x^2z+2h'xyz+b'y^2z+lxz^2+myz^2+nz^3=o.$ $\quad \ldots \quad \ldots \quad \ldots \quad (2)$

Or, symbolically, $u_0z^3+u_1z^2+u_2z+u_3=o \quad \ldots \quad (3)$

where u_0 is a constant and u_1, u_2, u_3, are homogeneous expressions of the first, second and third orders respectively in x and y.

140. We have seen, § 40, that a curve of the third order can have at most one double point, and no other multiple point. Hence according to their deficiencies, cubic curves may be divided into the following three fundamental species :—

(1) *Non-singular or anautotomic cubics*, which have no double points.

(2) *Nodal cubics*—in which the double point is a node, with two distinct tangents (real).

(3) *Cuspidal cubics*—in which the double point is a cusp.

... ...

By using the formulæ of § 97, we may calculate the Plücker's numbers for the three cases as follow :—

	$n=$	$\delta=$	$k=$	$m=$	$\tau=$	$i=$	$p=$
Case I ...	3	0	0	6	0	9	1
Case II ...	3	1	0	4	0	3	0
Case III ...	3	0	1	3	0	1	0

141. The trilinear equation of a cubic circumscribing the triangle of reference is

$$x^2u+y^2v+z^2w+kxyz=o \qquad \ldots \quad (1)$$

where u, v, w are linear functions of y, z ; z, x ; and x, y respectively and represent therefore the tangents at the vertices A, B, C respectively

If the vertex A be a double point on the cubic, the equation should contain no x^3 and x^2, and consequently it takes the form $xu_2+u_3=o$, where u_2 and u_3 are homogeneous functions of the second and third orders respectively in y and z. If further the curve passes through B and C, the equation cannot contain y^3 and z^3. Therefore the equation of a cubic circumscribing the fundamental triangle and having a double point at A is

$$xu_2+yz(my+nz)=o \quad \ldots \qquad \ldots \quad (2)$$

in which $u_2=o$ is the equation of the tangents at A. Hence the point A will be a node, a cusp, or a conjugate point on the curve, according as u_2 represents two real and distinct, coincident, or imaginary right lines.

142. If the vertex A is a point of inflexion on the curve, the tangent at A meets the cubic in three consecutive points. Now the equation of a cubic passing through A can be written as

$$x^2u_1+xu_2+u_3=o. \quad \ldots \qquad \ldots \quad (3)$$

If A is a point of inflexion, the tangent $u_1=o$ meets the curve in three consecutive points at A. Therefore, if u_1 is made equal to *zero* in the equation, *i.e.*, if y be eliminated between $u_1=o$ and the equation (3), the resulting equation should have x^3 as a factor, which requires that the coefficient of x should vanish, *i.e.*, u_2 should contain u_1 as a factor. Thus the equation of a cubic having a point of inflexion at A is

$$x^2u_1+xu_1v_1+u_3=o \quad \ldots \quad \ldots \quad (4)$$

143. We have proved in § 21 that if a cubic curve passes through eight points of intersection of two cubics, it must pass through the ninth also. As a particular case, we may prove the following theorem: If two right lines A and B meet a cubic in the points a, b, c and a', b', c' respectively, then the lines aa', bb', cc' respectively meet the cubic again in three other *collinear* points a'', b'', c''.

Let $u=o$ and $v=o$ be the equations of the two given lines A and B and let $u'=o$, $v'=o$, $w'=o$ be the equations of the lines aa'(A$'$), bb'(B$'$), cc'(C$'$) respectively. If $w=o$ be the equation of the line $a''b''$(C), then it must pass through the point c''. (Fig. 18).

The lines A$'$, B$'$, C$'$ constitute a cubic which intersects the given cubic in nine points, and the lines A, B, C make up a cubic passing through eight of these nine points. It must therefore pass through the ninth point c'' also. But this last point cannot lie on A or B, which already meet the cubic each in three points. Therefore it must lie on C.

Now, the equation of a cubic passing through the intersection of two cubics U$=o$ and V$=o$ is of the form U$-k$V$=o$. Therefore the equation of the given cubic can be written as $u'v'w'-kuvw=o$, since it passes through the intersections of the cubics uvw and $u'v'w'$.

Cor: From what has been said above, it follows that the equation of all cubics can be expressed in the form

$$uvw + ku'v'w' = o. \quad \dots \quad \dots \quad (5)$$

where u, v, w u', v', w', are linear functions of the variables, and therefore represent right lines. The equation represents a cubic passing through the nine intersections of (u, v, w) and (u', v', w').

144. If $u = v$, *i.e.*, if the lines A and B coincide, the equation of the cubic takes the form

$$u'v'w' + ku^2w = o. \quad \dots \quad \dots \quad (6)$$

The lines u', v', w' become tangents to the cubic at the three points where the line A meets it. Also, the three other points (a'', b'', c'') in which these tangents meet the cubic again lie on the line $w = o$. Hence we obtain the theorem:—

If a right line intersects a cubic in three points, the tangents at these points meet the cubic again in three other collinear points.

Definition: (1) The point a'', in which the tangent at any point a meets the cubic again, is called the **tangential** of the point a. This point is also called the "satellite point" of the tangent, (Cayley, A memoir on curves of the third order—Coll. Papers Vol. II, No. 146, p. 409).

(2) The line C on which lie the tangentials of three collinear points lying on a right line A is called the **satellite** of A. We may thus state the above theorem as follows:—

The tangentials of three collinear points are collinear.

145. As an application of the above properties of a cubic, we may prove the following theorem*: Having

* A. Cayley—Memoire sur les courbes du troisieme ordre—Journal de Mathematiques Pures et Appliqués (Liouville) tome IX (1884), or Coll. Papers. Vol I, No. 26. p. 184.

given a curve of the third order which passes through the six points of intersection of four right lines, the tangents to the curve at opposite points intersect on the curve in three collinear points. [Two points are said to be *opposite* when one is the intersection of two lines and the other of the remaining two].

Let 5 and 6 denote two points on the curve such that the tangents at these points intersect at a point $5'$ or $6'$ on the curve. Let 2 denote any other point on the curve and the lines (5, 2) and (6, 2) intersect the curve at the points denoted by 4 and 3 respectively. Then (5, 3) and (6, 4) intersect on the curve at the point 1. The tangents at 1 and 2, also the tangents at 3 and 4, intersect in two points on the curve which are collinear with the point $5'$. For the tangentials of 5, 3, 1 lie on $(5', 3', 1')$, and those of 6, 2, 3 are on the line $(6', 2', 3')$, *i.e.*, the line $(5', 1', 3')$. Hence the theorem. (Fig. 19).

146. *The three points in which a cubic intersects its asymptotes lie on a right line.*

We have defined that the asymptote of a curve is a tangent whose point of contact is at infinity. But a cubic curve has three asymptotes. Therefore, if we suppose that in § 143 the line $u=o$ is at infinity, *i.e.*, $u \equiv \mathrm{I}$, the equation of the curve becomes $u'v'w'+k\mathrm{I}^2w=o$, where $\mathrm{I}=o$ is the line at infinity, and u', v', w' are the tangents whose points of contact lie on $\mathrm{I}=o$, *i.e.*, $u'=o$, $v'=o$, $w'=o$ are the asymptotes. The form of the equation shows that the points in which these asymptotes meet the cubic again lie on the line $w=o$.

The straight line w which passes through the points of intersection of a cubic with its asymptotes is called the **satellite of the line at infinity.**

147. *The product of the perpendiculars drawn from any point on the curve on to the asymptotes is in a constant ratio to the perpendicular drawn from the same point on to the satellite of the line at infinity.*

This is only a geometrical interpretation of the equation of the previous article. The equation of a cubic whose asymptotes are u', v', w' and w is the satellite of the line at infinity is $u'v'w' = kI^2w = k'w$, where k' is a constant. Now, u', v', w' and w are proportional to the lengths of the perpendiculars drawn from any point of the curve on to those lines and hence the theorem.

148. *If two of the points of intersection of a line with a cubic be points of infle ion, the third must also be a point of inflexion.*

L t the points a and b be points of inflexion on the curve. If the line ab meets the curve at c, then c is also a point of inflexion.

Now, the tangent at a point of inflexion has a three-pointic contact with the curve. Consequently, the tangential of a point of inflexion coincides with the point itself. Thus the tangentials of a and b respectively coincide with them. Therefore the satellite of the line ab coincides with itself, and consequently the tangent at the third point c, in which the line ab cuts the cubic, has a contact of the second order, *i.e.*, the point c is a point of inflexion.

If we put $u' = v' = w'$ in the equation (5) of § 143, it becomes $uvw + ku'^3 = o$, which shows that the lines u, v, w have each a contact of the second order with the curve at the points where u' intersects it. Hence we obtain the theorem :—*If a cubic has three real points of inflexion, they lie on a right line.*

149. We have seen that the tangents to a cubic at three collinear points meet the cubic again in three other collinear points, or, what is the samething, that if tangents be drawn to a cubic from three collinear points a, b, c, on the curve, then the line joining the point of contact of *any* one of the tangents from a to the point of contact of *any* one of the tangents from b, passes through the point of contact of *any* one of the tangents from c. Now, from any point on a non-singular cubic four tangents can be drawn to it. Therefore the sixteen lines which join the four points of contact of tangents drawn from 'a' to those of the tangents from b, must pass through the four points of contact of the tangents drawn from c. Thus the twelve points of contact of these tangents lie on sixteen lines, three on each, and through each point there pass four of these sixteen lines.

From this it follows that, for a given line A, there is but one satellite to it; but to a given line A there correspond sixteen different lines, of which the given line is the satellite. Hence we obtain the theorem :—*A given line has only one satellite, but there are sixteen different lines of which it is itself the satellite.*

150. *The four points of contact of tangents drawn from any point A on a cubic are the vertices of a quadrilateral, the three diagonal points of which are the points of contact of the tangents drawn from the tangential point of A.*

Consider a line which intersects the cubic in the three points A, B, C. Let a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4 be the points of contact of tangents drawn from A, B, C respectively. Then these twelve points lie on sixteen different lines. Let the points A and B coincide. Then the points a_1, a_2, a_3, a_4 coincide with the points b_1, b_2, b_3, b_4 and one of the points c's (say c_4) coincides with A.

Thus we see that the line joining c_1, one of the points of contact of tangents from c, to a_1, one of the points of contact of tangents from A, must pass through one of the *other* points of contact of tangents from A, say a_2. Similarly the line c_1a_3 passes through a_4. Thus the sixteen lines reduce to six sides of the quadrangle $a_1a_2a_3a_4$, counted twice, and the four tangents at these points. Hence the intersection of a pair of opposite sides is one of the points c_1, c_2, c_3, and the tangents at the vertices a_1, a_2, a_3, a_4 meet the curve at the same point A (c_4), *i.e.*, c_1, c_2, c_3 are the diagonal points of the quadrilateral.

151. From this we easily deduce the truth of the theorem:—*If two tangents be drawn from any point A on a cubic, the tangent at the third point in which the chord of contact meets the cubic cuts the tangent at A at a point on the curve.*

We may analytically prove the theorem as follows:—

If we take the two tangents and their chord of contact as the sides of the triangle reference, the equation of the cubic must be of the form $yz(lx+my+nz)+x^2(\mathrm{B}y+\mathrm{C}z)=o$, where $\mathrm{B}y+\mathrm{C}z=o$ is the tangent at A and $lx+my+nz=o$ is the tangent at the point where the chord of contact $x=o$ meets the curve. These two lines intersect on the curve.

152. *The chords of contact of tangents drawn from any point of a cubic are harmonic conjugates of the tangent to the curve at their intersection and the line joining the intersection with the point.*

Let the line C be the satellite of any line A, the tangents at the points on A being L, M, N. Then the equation of the curve can be written as $\mathrm{LMN}-\mathrm{A}^2\mathrm{C}=o$, if $\mathrm{A}=o$ and $\mathrm{C}=o$ represent those lines. Let B be any other line of which C is the satellite, so that B passes through the point of contact of N and those of two other tangents L′ and M′ respectively which meet C in the

points where L and M respectively meet it. Then the equation of the curve may again be written as $L'M'N - B^2C = o$. Thus we obtain the identity $N(LM - L'M') = (A^2 - B^2)C$. The right-hand side represents three lines $A \pm B$ and C, therefore the left-hand side must also represent three right lines. Now the line N must be one of $A \pm B$, and C must be one factor of $LM - L'M'$, which is the line (LL', MM'), and the other factor is the line $(LM', L'M)$ which is $A \mp B$. Therefore when C is a tangent, so that L, M, L′ and M′ meet C at the same point on the cubic, one of $A \mp B$ becomes the line joining the point of contact of C with that of N. But A, B, $A \pm B$ form a harmonic pencil. Hence the theorem. (Fig. 20.)

153. *Any line drawn through any point A on a cubic is cut harmonically in the two points P and Q where it meets the cubic again, and the two points L and M where it meets a pair of chords joining the points of contact of tangents drawn from A.*

Let a_1, a_2, a_3, a_4 be the four points of contact of tangents drawn from any point A on the curve. (Fig. 21.) Then the lines joining a_1, a_2 and a_3, a_4 intersect at a point C_1 on the curve. Let a line through A intersect the cubic in P and Q and the tangent at C_1 at D, and the chords of contact at L and M respectively. By the previous theorem, $C_1(LAMD)$ is a harmonic pencil *i.e.*, LM is a harmonic mean between LA and LD.

$$\therefore \frac{1}{LA} + \frac{1}{LD} = \frac{2}{LM}. \quad \ldots \quad \ldots \quad (1)$$

By Maclaurin's another theorem (§ 53), since any line through L intersects the curve at P, A, Q and the tangents at three points (collinear with L) in the three

points A, A and D, we have

$$\frac{1}{LP}+\frac{1}{LA}+\frac{1}{LQ}=\frac{1}{LA}+\frac{1}{LA}+\frac{1}{LD}$$

$$i.e. \quad \frac{1}{LP}+\frac{1}{LQ}=\frac{1}{LA}+\frac{1}{LD}=\frac{2}{LM}.$$

$\therefore$ PQLM are harmonic.

154. The theorem of § 19 can be applied to the case of the cubic when $m=3$, and it then takes the form :—
Every curve of the nth degree which passes through $3n-1$ fixed points on a cubic passes through one other fixed point on the curve.

This can be proved very easily with the help of the theory of residuation. Let the group of $3n-1$ points be denoted by P. Describe two curves of the nth degree through these points P. Let them intersect the cubic in the points Q and Q′ respectively. Then $[P+Q]=o$ and $[P+Q']=o$. $\therefore$ $[Q]=[Q']$ *i.e.* Q and Q′ coincide.

If $n=1$ or $n=2$, the theorem asserts nothing, as only one curve can be drawn through $3n-1$ points. Hence the truth of the theorem is manifest when n is greater than two.

155. *If any conic be described through four fixed points on a cubic, the chord joining the two remaining intersections of the conic with the cubic will pass through a fixed point on the curve.*

Let the system of four points be denoted by P and describe two conics through P, intersecting the cubic in the two pairs of points denoted by Q and Q′ respectively. If the line joining the points Q meet the cubic again in R and that joining the points Q′ in R′, then R and R′ must coincide. For, we have $[P+Q]=o$ and $[P+Q']=o$.

$$\therefore \ [Q]=[Q']$$

Also $[Q+R]=o$ and $[Q'+R']=o$;

$\therefore\ [R]=[R']$ *i.e.* R and R′ coincide.

Second Proof:—

Let A, B, C, D be the four given points and let the sides AB, BC, CD, DA be respectively denoted by x, y, z, w. Then the equation of the cubic is $xzu=ywv$, where u and v are linear functions of the variables. The equation of the conic may be written as $xz=kyw$. Combining these two equations we obtain $v=ku$, which is the equation of the line through the two remaining intersections of the cubic and the conic. But this line always passes through the fixed point $u=v=o$ on the cubic, for all conics given by different values of the parameter k.

156. *A hexagon is inscribed in a cubic. If two pairs of opposite sides meet in points on the cubic, the third pair also intersects on the curve.*

Let the first, third and fifth sides of the hexagon be denoted by P, Q, R and the second, fourth and sixth sides by P′, Q′, R′. Let the sides P, P′ intersect at A, and Q, Q′ intersect at B on the cubic. Then the third pair R, R′ must intersect at a point on the curve.

For, consider the following cubics through the six vertices of the hexagon and the two points A, B :—

(1) the given cubic,

(2) the cubic consisting of the lines P, Q, R,

(3) „ „ „ „ „ P′, Q′, R′.

Therefore they must pass through a ninth common point *i.e.*, R and R′ intersect the cubic at the same point C on the curve.

Cor: If R and R′ are tangents to the cubic, the theorem becomes:—

If the opposite sides of a quadrilateral inscribed in a cubic intersects on the curve, the tangents at the opposite vertices meet at the same point on the cubic i.e. the opposite vertices have the same tangential point.

157. The theorem of §155 is an immediate consequence of a theorem in residuation, namely, a pencil of lines is equivalent to a pencil of conics *i.e.*, a pencil of lines is projectively related to a pencil of conics.

Let $a\mathrm{L}+\lambda b\mathrm{M}=0$ be a pencil of lines, whose vertex is the point $\mathrm{L}=\mathrm{M}=0$. A pencil of conics, projectively related with it, is $\phi+\lambda\psi=0$. If we eliminate λ between these two equations, we obtain

$$f\equiv a\mathrm{L}\psi-b\mathrm{M}\phi=0.$$

which represents a cubic curve passing through the four points of intersection of ϕ and ψ, the points where L intersects ϕ, and M intersects ψ, and the point $\mathrm{L}=\mathrm{M}=0$.

It will be noticed that the point $\mathrm{L}=0$, $\mathrm{M}=0$ is the coresidual of the system of four points common to ϕ and ψ.

Hence we obtain the following:—

*If four points on a cubic be taken as the base of a pencil of conics, the coresidual point is the vertex of a pencil of lines, projective with the pencil of conics, the two pencils together generate the curve.**

This property at once suggests the following method for constructing a cubic through nine given points:—

The nine points must be independent, *i.e.*, they should not be the intersections of two other curves of the third

* Chasles—Construction de la courbe du troisieme ordre determineé par neuf points. (Comp. Rendus, May, 1853.)

order, for in that case an infinite number of such curves can be described through them, and the problem becomes indeterminate. It is sufficient to indicate the method how the solution may be effected. With any four of these points as base, we have to describe a pencil of conics and then determine the point M coresidual to these four. This point is obtained as the fourth intersection of two conics three of whose other intersections are given.* Thus having determined M, we may easily establish a projective relation between a pencil of lines through M and a pencil of conics through the four points, such that there is a (1, 1) correspondence between them and consequently two other points are obtained by the intersection of the corresponding elements of the two pencils.

158. Having eight points given, to construct the ninth point which, together with these eight points, forms the nine intersections of two cubics.

The solution of this problem depends upon the construction of the point coresidual to any four of these points, corresponding to any two different curves passing through them, and also upon the application of the following theorem which can be easily demonstrated :—

The locus of points, coresidual to the four base points of a system of curves of the third order, on different curves of the system is a conic passing through the five other base points.

Let the system of cubics be given by $xzu=kywv$ and a conic through four of the nine points be given by $xz=k'yw$. Then the coresidual for a particular curve is given by $u=\frac{k}{k'}v$. But the conic through the remaining five points,

---

* C. A. Scott—loc. cit. §230.

is $uv=o$, and therefore the line $u=\frac{k}{k'}v$ passes through $u=v=o$, a point on $uv=o$.

158(a). A cubic with a double point can be generated by a pencil of conics and a pencil of lines, projective with it, whose vertex is at a base-point of the pencil of conics.

Let $y=\lambda x$ be the pencil of lines; then a conic through $x=y=o$ is of the form $Lx+My=o$, where L and M represent right lines. A projective pencil has the equation $(Lx+My)+\lambda(L'x+M'y)=o$. The elimination of λ gives $L'x^2+(M'-L)xy-My^2=o$, which is a cubic with a double point.

158(b). A general n–ic can also be generated by projective pencils of curves. It is very easy to show this analytically.

Let $l+m=n$. Then the two projective pencils of curves, of orders l and m respectively, $C_l+\lambda C'_l=o$ and $C_m+\lambda C'_m=o$ will generate the curve of the nth order, *viz.*, $C_l C'_m - C'_l C_m=o$.

It is somewhat difficult to show this for a given curve or for a system of $\frac{1}{2}n(n+3)$ given points. We shall show this for a pencil of lines and a pencil of curves of $n-1$ order.

Through any point P of the given n–ic we draw a pencil of lines $L+\lambda M=o$, which determines a singly infinite group of $(n-1)$ points. Through one group G_{n-1} of this, an $(n-1)$–ic C_{n-1} can be drawn. This intersects the n–ic in $n(n-1)-(n-1)=(n-1)^2$ other points. Through these $(n-1)^2$ points, then, there passes a pencil $C_{n-1}+\lambda C'_{n-1}$ of $(n-1)$–ics, which intersects the n–ic at the same group of points as the pencil of lines. The two pencils then generate the curve. The equation of the curve is then $C_{n-1}.M-C'_{n-1}L=o$.

159. *The locus of a point, such that the lines joining it to three fixed points intersect three fixed lines in three collinear points, is a curve of the third order.**

Let $P(x, y, z)$ be the variable point, (Fig. 22) and $a(a_1, a_2, a_3)$, $b(b_1, b_2, b_3)$ and $c(c_1, c_2, c_3)$ be three fixed points, with reference to the three fixed lines as the sides of the triangle of reference, then the equation of the line Pa is—

$$X(ya_3-za_2)+Y(za_1-xa_3)+Z(xa_2-ya_1)=0 \qquad \dots \quad (1)$$

This meets the side $L(X=0)$,

where
$$\frac{Y}{xa_2-ya_1}=\frac{-Z}{za_1-xa_3} \qquad \dots \quad (2)$$

Similarly, the other two points of intersection are given by

$$\frac{X}{xb_2-yb_1}=\frac{Y}{0}=\frac{-Z}{yb_3-zb_2} \qquad \dots \quad (3)$$

and
$$\frac{X}{zc_1-xc_3}=\frac{-Y}{yc_3-zc_2}=\frac{Z}{0} \qquad \dots \quad (4)$$

$\therefore$ If these three points are collinear, we must have

$$\begin{vmatrix} 0 & (xa_2-ya_1), & -(za_1-xa_3) \\ xb_2-yb_1, & 0 & -(yb_3-zb_2) \\ zc_1-xc_3, & -(yc_3-zc_2) & 0 \end{vmatrix}=0 \qquad \dots \quad (5)$$

From equation (5) it follows that the locus is of order three and passes through the points a, b, and c and through the vertices of the triangle of reference. It can be shown that the curve passes also through the points where the lines bc, ca, ab respectively meet the sides.

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* H. G. Grassmann—Die lineale Ausdehnungslehre, 1844, Leipzig.

160. *The tangentials of the three points at which a conic has simple contact with a curve of the third order are in one right line.*

Let A, B, C be the three points of contact and A′, B′, C′ their tangentials respectively. Then we have three curves of the third order which pass through eight common points, namely, the three points of contact, each counted as two, and the points A′, B′ :—

(1) The given curve of the third order,

(2) The three tangents at A, B and C ;

(3) The conic having simple contact with the cubic at A, B, C together with the line A′B′.

These three curves must therefore pass through a ninth common point C′. Now in the cubic (3), the conic cannot pass through C′ which already meets the cubic in six points. Hence C′ lies on A′B′.

Note : Hence if A and B are given, we can determine C by the following simple construction : Draw the tangents to the cubic at the points A and B, and let these tangents intersect the curve again in A′ and B′ respectively. The line A′B′ intersects the cubic again at a point C′. The points of contact of the tangents drawn from C′ are the required points C. But in general four tangents can be drawn from C′ to the curve, and the point of contact of any one of these tangents will be the required point C. But three of these points will give the solution of the problem; for AB intersects the cubic at a third point D, the tangent at which also passes through C′. In fact the line AB taken twice may be regarded as a conic having simple contact with the cubic at the points A, B and D. Consequently there are three different systems of doubly infinite number of conics which have simple contact with the curve at three different points.

161. If two of the points A, B, C (say A, B) coincide, the conic has a four-pointic contact at A and a simple contact at C. In consequence, the points A′ and B′ coincide and C′ is the tangential of the point A′, or as we say, C′ is the second tangential of A.

Hence to construct a conic having a four-pointic contact at A and a simple contact elsewhere, we proceed as follows :—Draw the tangent at A to the cubic and let A′ be the tangential of A and let C′ be the tangential of A′ *i.e.*, the second tangential of A. Then the four points of contact of the tangents drawn from C′ to the curve will be the required points C. But one of these points is A′, which is the tangential of A. Hence the other three points of contact will give the solution of the problem. Thus three conics can be drawn having a four-pointic contact at any given point of a cubic and a simple contact elsewhere.

162. *If a conic osculates a cubic curve at two distinct points, the chord of contact intersects the cubic again in a point of inflexion.*

Consider a conic which has a contact of the second order with a cubic at the two points A and B. The conic passes through three consecutive points at A and at B. Let A′B′ and A″B″ be two other lines consecutive to AB, so that they pass through these consecutive points at A, and B. These lines intersect the cubic in three other consecutive points C, C′, C″.

We have then the following three cubics passing through the eight points A, A′, A″; B, B′, B″; C, C′ :—

(1) The given cubic,

(2) The cubic consisting of the three lines A′B′, A B, A″B″;

(3) The conic and the line CC′.

Therefore they must pass through a ninth common point C'', *i.e.* the line CC' must pass through C''. Thus three consecutive points C, C', C'' are collinear, or, in other-words, the point C in which the line AB cuts the cubic again is a point of inflexion on the cubic.

The theorem of this article may be stated in a different form :—

Every chord drawn through a point of inflexion intersects a cubic in two other points such that a conic can be described having a three-pointic contact with the curve at those points.

Again, since a cubic has nine points of inflexion, there are nine different systems of conics which have a contact of the second order at two points of a cubic curve.

163. *There are twenty-seven conics which have a contact of the fifth order or a six-pointic contact with a non-singular cubic. These points are the twenty-seven points of contact of the tangents drawn from the nine points of inflexion.*

Suppose the two points A and B in the preceding article coincide. Then the conic has a six-pointic contact with the curve at A, which becomes the point of contact of a tangent drawn from the point of inflexion. Hence, we see that the points of contact of the tangents drawn from a point of inflexion are the points, where a conic can be drawn having a six-pointic contact with the curve. Now there are nine points of inflexion and from each three tangents can be drawn to the curve. Hence there are twenty-seven points of contact of such tangents and each of these points gives a solution of the problem.

Definition: A *sextactic point* on a curve is a point the osculating conic at which passes through six consecutive points or has a six-pointic contact at that point.

The above theorem may therefore be stated as:—*The points of contact of the tangents drawn from a point of inflexion are sextactic points on the cubic.*

Let P be a point of inflexion and A be the point of contact of a tangent drawn from P. Then $[3P]=o$ and $[P+2A]=o$, or, $[3P+6A]=o$. Hence by the subtraction theorem, $[6A]=o$, or, A is a sextactic point.

164. *If a conic osculates a cubic at two distinct points, one of which is a sextactic point, then the other point must also be a sextactic point.*

This is easily proved by residuation:

Let P and Q be the two points of osculation.

Then, $[3P+3Q]=o$. But P being a sextactic point, $[6P]=o$. Therefore, by the theorems of multiplication and subtraction we obtain $[6Q]=o$, which shows that six consecutive points at Q are the complete intersections of the cubic with a curve, which is evidently a conic.

CHAPTER XI.

Harmonic Properties of Cubic Curves.

165. Every point has a polar conic and a polar line with respect to a cubic. The polar conic is the first polar which, therefore, passes through the points of contact of the six tangents drawn from the point to the curve.

The equation of the polar conic of a point (x', y', z') is

$$x' \frac{df}{dx} + y' \frac{df}{dy} + z' \frac{df}{dz} = o,$$

and that of the polar line is

$$x \frac{df}{dx'} + y \frac{df}{dy'} + z \frac{df}{dz'} = o.$$

The equation of the polar conic in the expanded form may be written, as before, as $a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy = o$, where $a', b', c' \ldots$ represent the second differential co-efficients in which (x', y', z') have been substituted for (x, y, z).

166. *Every right line has four poles with respect to a non-singular cubic.*

Let A (x_1, y_1, z_1) and B (x_2, y_2, z_2) be any two points on the line and let P (x', y', z') be a pole.

Now, the conditions that the polar line of P passes through A and B respectively are

$$U' \equiv x_1 \frac{df}{dx'} + y_1 \frac{df}{dy'} + z_1 \frac{df}{dz'} = o \quad \ldots \quad (1)$$

$$V' \equiv x_2 \frac{df}{dx'} + y_2 \frac{df}{dy'} + z_2 \frac{df}{dz'} = o \quad \ldots \quad (2)$$

Equations (1) and (2) show that the polar conics of A and B pass through P. Hence P is one of the four points in which the polar conics of A and B intersect, and therefore a right line AB has four poles with respect to a cubic.

Again, the polar conic of any point C on AB, passes through these four poles. For if C be the point $(x_1+\lambda x_2,\ y_1+\lambda y_2,\ z_1+\lambda z_2)$, the polar conic of C is $U+\lambda V=o$, which evidently passes through the intersections of the polar conics of A and B.

Thus *the polar conics of points on a right line form a pencil of conics through the four poles of the line.*

Conversely, any conic through the four poles of a line is the polar conic of some point on the line.

167. Thus we see that every right line has only *four* poles with respect to a non-singular cubic. These are the points of intersection of the polar conics of two points on the line. If the cubic has a node, all the polar conics pass through the node and therefore the two polar conics intersect in only *three* other points, which are the poles of the line. In the case of a cuspidal cubic, the polar conic passes through the cusp and touches the cuspidal tangent. Thus the two polar conics intersect in *two* other points only, which are the poles of the line. If the cubic reduce to three right lines, all polar conics pass through the vertices of the triangle formed by them, and every right line has only *one* pole.

168. The three pairs of right lines through the four poles of a line with respect to a non-singular cubic are the polar conics of three points L, M, N on the line. These three points must therefore lie on the Hessian of the cubic. Hence L, M, N are the points in which the

line intersects the Hessian, which therefore is a curve of the third order.

We shall now prove the that *the Hessian is the envelope of a line two of whose poles coincide.*

Let the four poles of the line AB be P, Q, R and S, and let the pairs of lines PQ, RS ; PS, QR; and PR, QS, which constitute the polar conics of the three points L, M, N on AB, intersect in the points O, E, F respectively. (Fig. 23.) Then the polar conics of O, E, F break up into pairs of right lines intersecting at L, M, N respectively.

Suppose the two poles S and R coincide. Then O and F coincide with R and therefore the corresponding points M and N must coincide. Thus the line LMN becomes the tangent to the Hessian at the point M, which corresponds to O or F. The point L corresponds to E. Hence if a line touch the Hessian, two of the points O, E, F coincide at one of the poles, which corresponds to the point of contact. This is true for every tangent to the Hessian and hence the truth of the theorem follows.

Thus the four poles of a tangent to the Hessian are the points R counted twice and the points P and Q. But P and Q become the points of contact of PF and QF with their envelope, which, as we shall see later, is the Cayleyan.

169. *If three tangents be drawn to a cubic from a point of inflexion, their points of contact lie on a right line.*

The equation of a cubic having the vertex A of the triangle of reference as a point of inflexion is—

$$f \equiv x^2 u_1 + x u_1 v_1 + u_3 = 0.$$

The first polar or the polar conic of A is—

$\frac{df}{dx}=u_1(2x+v_1)=0$ *i.e.*, the polar conic breaks up into two right lines, one of which $u_1=0$ is the tangent at A, and the other line $2x+v_1=0$ passes through the points of contact of the tangents drawn from A.

Definition: The line $(2x+v_1)=0$, which passes through the points of contact of the tangents drawn from a point of inflexion, is called the *Harmonic Polar* of the point of inflexion.

Note. This line is called the harmonic polar simply to distinguish it from the ordinary polar line of the point A, which in the present case is the inflexional tangent at A.

170. *Every chord drawn through a point O on a cubic is cut harmonically by the curve and the polar conic of the point O.*

Let O be the origin and the equation of the cubic be $bx+cy+dx^2+exy+fy^2+gx^3+\ldots\ldots=0$.

Now the polar conic of O is given by $\frac{df}{dz}=0$;

i.e., $2(bx+cy)+dx^2+exy+fy^2=0$. Let the line $\frac{x}{l}=\frac{y}{m}=r$ intersect the cubic in P and Q and the polar conic in A.

Then, substituting lr and mr respectively for x and y in these two equations, we obtain

$$\frac{\mathrm{OP}+\mathrm{OQ}}{\mathrm{OP}\,.\,\mathrm{OQ}}=-\frac{dl^2+elm+fm^2}{bl+cm}=\frac{1}{\mathrm{OP}}+\frac{1}{\mathrm{OQ}}.$$

and $\mathrm{OA}=-\dfrac{2(bl+cm)}{dl^2+elm+fm^2}.$

$\therefore\ \dfrac{1}{\mathrm{OP}}+\dfrac{1}{\mathrm{OQ}}=\dfrac{2}{\mathrm{OA}}$ *i.e.*, the points O, P, A, Q form a harmonic range.

171. If the point O is a point of inflexion on the curve and the axis of y the tangent at it, we must have $c=o$, and $f=o$, and the equation becomes $bx+x(dx+ey)+gx^3+\ldots=o$.

The polar conic is $(2bx+dx^2+exy)=o$, or, $x(2b+dx+ey)=o \therefore 2b+dx+ey=o$ is the harmonic polar of O.

Thus we obtain the theorem :—*Every chord drawn through a point of inflexion O on a curve is divided harmonically by the curve and the harmonic polar*; or in other words,

If radii vectores be drawn through a point of inflexion, the locus of the harmonic means will be a right line (*the harmonic polar*).

172. *If a chord drawn from any point A of a cubic intersect the curve again in B and C and the polar conic of A in D, then the tangents to the cubic at B and C and the tangent to the polar conic at D all meet at the same point.*

Consider another line drawn through A which intersects the cubic in B′ and C′ and the polar conic of A in D′. Now, by §170, the ranges (A B D C) and (A B′ D′ C′) are harmonic. Let the lines BB′ and CC′ intersect in O. Join OD and OD′. Therefore OD and OD′ are each harmonic conjugate of OA with respect to OBB′ and OCC′. Hence OD and OD′ coincide *i.e.* the line DD′ passes through O. Now if ABC and AB′C′ coincide, BB′ and CC′ become tangents to the cubic and DD′ becomes tangent to the polar conic. Hence the theorem.

173. *The node of a nodal cubic is the pole of the line joining its three points of inflexion.*

The equation of a cubic having three points of inflexion can be written as $u'v'w'=u^3$. Now, choose $u,'v,'w'$ as

the sides of the triangle of reference and let $u \equiv lx + my + nz = o$.

Thus the equation becomes $xyz = (lx+my+nz)^3 = u^3$.

If this has a double point $(x,'y,'z')$, the first differential coefficients must vanish at that point *i.e.* we must have

$$y'z' = 3lu'^2,\ z'x' = 3mu'^2,\ x'y' = 3nu'^2,$$

where $\qquad u' = lx' + my' + nz'. \quad \therefore\ lx' = my' = nz'. \qquad \ldots (1)$

The polar line of $(x,'y,'z')$ is—

$$x(y'z' - 3lu'^2) + y(z'x' - 3mu'^2) + z(x'y' - 3nu'^2) = o. \ \ldots (2)$$

Now, from (1) $\dfrac{y'z'}{l} = \dfrac{z'x'}{m} = \dfrac{x'y'}{n}$;

or, $$\frac{y'z' - 3lu'^2}{l} = \frac{z'x' - 3mu'^2}{m} = \frac{x'y' - 3nu'^2}{n}.$$

Therefore the equation (2) becomes $lx + my + nz = o$, which proves the proposition.

174. A point of inflexion on a cubic possesses, with regard to its harmonic polar, properties analogous to those of poles and polars in conic sections. We shall first of all prove that *if two right lines be drawn through a point of inflexion to meet a cubic in four points and the extremities be joined directly and transversely, the two points of intersection lie on the harmonic polar.*

This follows immediately from the harmonic properties of a quadrilateral. For, let A be a point of inflexion and let AB and AC meet the cubic in B, D and C, E respectively. Let BE and CD meet in G and BC, DE intersect at H. Then GH is the harmonic polar of A. For, if GH meets AB and AC in L and M respectively, (A B L D) and (A C M E) are harmonic. Hence GH is the locus of harmonic means between AB and AD. (Fig. 24).

175. The above theorem easily leads to the following:—If a line intersects a cubic in three points A, B, C, then the lines joining A, B, C to the point of inflexion O meet the cubic again in three collinear points A,′ B,′ C′ and the two lines ABC, A′B′C′ meet the harmonic polar in the same point.

If however A, B, C coincide, then A,′ B,′ C′ also coincide, and it follows that the three points of inflexion O, A, A′ are collinear, and the tangents at A and A′ meet on the harmonic polar of O *i.e. the inflexional tangents at two of three collinear points of inflexion meet on the harmonic polar of the third.*

Let the three points of inflexion lie on the line $lx+my+nz=o$. The equation of the cubic may be written as $xyz=(lx+my+nz)^3=u^3$. Now, one point of inflexion is $x=o$, $lx+my+nz=o$ *i.e.* the point $(o,-n, m)$. The polar conic of this is $-n(xz-3mu^2)+m(xy-3nu^2)=o$ or $x(my-nz)=o$. ∴ The harmonic polar is $my-nz=o$, which evidently passes through the intersection of y and z, the other two inflexional tangents.

176. As a particular case of the above theorem we may deduce the following :—*Tangents at the extremities of any chord drawn through a point of inflexion intersect on the harmonic polar.*

This follows by considering that the lines AB and AC coincide, when BC and DE become tangents at B and D.

Again, if we consider that B and D coincide, *i.e.*, AB becomes a tangent, then the harmonic polar passes through the point of contact of AB. For, (ABLD) is harmonic and when B coincides with D, it must coincide with L.

Hence we obtain the theorem that *the harmonic polar passes through the points of contact of the tangents drawn from the point of inflexion.*

Now, since the harmonic polar intersects the cubic in three points, three tangents can be drawn from a point of inflexion, and their points of contact lie on a right line. (§169).

177. *The harmonic polars of the three points of inflexion which lie on a right line pass through the pole of that line with regard to the triangle formed by the inflexional tangents.*

Let the equation of the cubic be $xyz=(lx+my+nz)^3 \equiv u^3$ (say). Then x, y, z are the inflexional tangents and the points of inflexion lie on $lx+my+nz \equiv u=0$. The harmonic polars of the three points of inflexion are respectively $my-nz=0$, $nz-lx=0$, $lx-my=0$, which evidently meet at the point $lx=my=nz$, *i.e.*, at the point $(1/l, 1/m, 1/n)$ which is the pole of the line $lx+my+nz=0$ with regard to the triangle of reference.*

178. *Each harmonic polar passes through the node of a nodal cubic.*

If $xyz=(lx+my+nz)^3$ be the equation of the cubic, the harmonic polars of the points of inflexion are $my-nz=0$, $nz-lx=0$, $lx-my=0$, which pass through the point $lx=my=nz$. If the cubic has a node, we have $lx=my=nz$, at the node. Hence the theorem.

Note : Only one tangent can be drawn from a point of inflexion to a nodal cubic. Therefore the harmonic polar is the line joining the node to the point of contact of the tangent. When the curve has a cusp, the harmonic polar is the cuspidal tangent.

179. Inflexional Triangles :

It has been shown that a non-singular cubic has nine points of inflexion. These nine points of inflexion have

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* Scott— loc. cit. §. 23.

definite positions relative to one another. The line joining any two of them passes through a third. Hence it follows that through each of them there pass four lines, each of which passes through two of the remaining eight points of inflexion. These lines are called *inflexional lines.* Taking into account all such lines, it is seen that in actual arrangement each line is counted thrice, and consequently there exist only $\frac{1}{3}.9.4=12$ such lines. Let us denote the points of inflexion by the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, and the inflexional line through three points of inflexion i, j, k by the symbol L_{ijk}, for example, L_{123} denotes the line through 1, 2, 3. Thus the three lines L_{123}, L_{456}, L_{789} are such that they contain all the nine points of inflexion. They form a triangle which is called an *inflexional triangle.* We can easily determine the number of such triangles. Through each point of inflexion there pass four inflexional lines, each of which must be a side of one and only one triangle. But as the point of inflexion must lie in each triangle, there will be only *four* triangles, such that each contains all the nine points of inflexion. Thus we obtain the following theorem :—

The nine points of inflexion lie, three by three, on twelve lines and these lines can be grouped, three by three, to form four inflexional triangles such that each group contains all the nine points of inflexion.

We have seen that the points of inflexion on a curve are the points where the Hessian intersects it. In the case of a cubic curve, the Hessian is also a cubic, and therefore, the nine points of inflexion are the points through which there pass two cubics, and hence an infinite number. Therefore a pencil of cubics passes through these nine points of inflexion. To this pencil also belong the four inflexional triangles as degenerate cubics. For, consider that the Hessian coincides with the primitive curve.

In that case all points on the curve are points of inflexion, which is possible only when the cubic consists of right lines. In fact each inflexional triangle constitutes a degenerate cubic passing through the nine points of inflexion.

180. Configuration of the Harmonic Polars :

It is easy to show that the nine harmonic polars are the same for all curves of the pencil passing through the nine points of inflexion. We shall therefore obtain polars of the points of inflexion with respect to all these curves, if we consider an inflexional triangle as the primitive curve and construct the polar conic of a point of inflexion with respect to this triangle. The polar conic in question consists of the side of the triangle which passes through the point, and of the polar line of the point of inflexion with respect to two other sides of the triangle. If therefore we join the point to the vertex of one of the four inflexional triangles, opposite to the side on which it lies, and find the fourth harmonic to this line and the two sides of the triangle, this last line is the harmonic polar. It follows therefore that the harmonic polars of three collinear points of inflexion are concurrent in the vertex opposite to the inflexional line. Thus, through a vertex of each inflexional triangle there pass three harmonic polars and consequently each harmonic polar passes through one vertex of each of the four inflexional triangles. We thus obtain the theorem :

The twelve vertices of the inflexional triangles are situated, four by four, on the nine harmonic polars. The harmonic polars of three points of inflexion intersect at a vertex of the triangle corresponding to the inflexional line ; to the sides of a triangle there correspond the opposite vertices.

Thus we see that the nine harmonic polars form a dual system to the nine points of inflexion. They determine four triangles, through the vertices of each passing all the nine harmonic polars. These triangles are identical with the inflexional triangles, and each harmonic polar passes through one vertex of each of these triangles. In fact, to the pencil of order-cubics through the points of inflexion there corresponds, by the principle of duality, a system of class-cubics* with these nine harmonic polars as common cuspidal tangents. This system of class-cubics, as we shall see later, is the system of Cayleyan curves of the cubics in the pencil.

* A class-cubic is a curve of the third class, but not necessarily of the third order. Vide—C. A. Scott—loc. cit. § 67.

CHAPTER XII.

CANONICAL FORMS.

181. The general equation of a ternary cubic may be written as $u_3+u_2z+u_1z^2+u_0z^3=0$ and contains ten arbitrary constants. The triangle of reference can be so chosen that six of these constants vanish, for the change of the triangle of reference involves six arbitrary parameters, each side being defined by two independent conditions. Thus the trilinear equation of a certain class of cubic curves can be transformed into the form $ax^3+by^3+cz^3+lxyz=0$. Now, if we put $X=a^{\frac{1}{3}}x$, $Y=b^{\frac{1}{3}}y$, $Z=c^{\frac{1}{3}}z$, the equation takes the form $X^3+Y^3+Z^3+kXYZ=0$. This is called the *canonical form* of the equation and is due to Hesse.

This transformation is possible only in the case of a cubic whose discriminant does not vanish *i.e.*, in the case of a non-singular cubic. It is proved in works on Algebra* that the equation of such a cubic can be reduced to the form $x^3+y^3+z^3+6mxyz=0$.......................(1) where x, y, z are linear functions of the variables and therefore may be regarded as the co-ordinates of a point referred to a new triangle whose sides, referred to the original triangle, are $x=0$, $y=0$, $z=0$. The number of independent constants involved in this equation is $3\times3+1=10$, and therefore is the same as that of the co-efficients in the general equation. Thus we have an *a priori* indication of the possibility of such transformation.

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* E. B. Elliot—Algebra of Quantics, §229.

182. In order to transform an equation to its canonical form, we must first ascertain the new triangle of reference, which can be assumed to have some geometrical relation with the given curve. We shall first investigate this relation.

The equation $x^3+y^3+z^3+6mxyz=0$ can be written as

$$x^3+y^3-8\,m^3z^3-3xy\,(-2mz)=-(1+8m^3)z^3\,;$$

or, $(x+y-2mz)\,(\omega x+\omega^2y-2mz)\,(\omega^2x+\omega y-2mz)$

$$=-(1+8m^3)z^3,$$

where ω is an imaginary cube root of unity.

Now, if $1+8m^3\neq 0$, the equation can be put into the symbolic form $P.Q.R=z^3$, where P, Q, R are linear functions of the variables. From this form of the equation, it follows at once that P, Q, R are the inflexional tangents at the three points where $z=0$ cuts the curve. Hence we see that $z=0$ cuts the curve in three of its points of inflexion. Similarly, since the equation is symmetrical in x, y and z, we find that $x=0$ and $y=0$ also cut the curve each in three points of inflexion.

Thus the sides of the new triangle of reference are the three lines on which the nine points of inflexion lie, three by three.

Hence in order to reduce the equation of a cubic to its canonical form, we must take the three lines containing the nine points of inflexion, three by three, as forming the sides of the triangle of reference; *i.e.*, one inflexional triangle is taken as the triangle of reference.

183. The points where $x=0$ cuts the curve $x^3+y^3+z^3+6mxyz=0$ are determined by the equations $x=0$ and $y^3+z^3=0$ *i.e.*,

$$(y+z)\,(y+\omega z)\,(y+\omega^2z)=0.$$

Since these co-ordinates are independent of m, it follows that if m is a variable parameter, all cubics included in the canonical form cut the three sides of the triangle of reference in the same nine points, three of which are real and six imaginary.

We have seen that if $1+8m^3 \neq 0$, these points are points of inflexion on the cubic. Hence all cubics given by the canonical form, where m is a variable parameter, have the same nine points of inflexion.

184. Co-ordinates of the nine points of inflexion :

The equation of the cubic can be written in the three following forms :—

$$(-2mx+y+z)\,(-2mx+\omega^2 y+\omega z)\,(-2mx+\omega y+\omega^2 z)$$
$$+(1+8m^3)x^3=0\,;$$
$$(x-2my+z)\,(\omega x-2my+\omega^2 z)\,(\omega^2 x-2my+\omega z)$$
$$+(1+8m^3)y^3=0\,;$$
$$(x+y-2mz)\,(\omega x+\omega^2 y-2mz)\,(\omega^2 x+\omega y-2mz)$$
$$+(1+8m^3)z^3=0\,;$$

The co-ordinates of the nine points of inflexion are therefore obtained as follow :—

$$x=0,\ \begin{cases} y+z=0, & i.e.\ (0,1,-1) \quad \ldots \quad \ldots \quad (1) \\ y+\omega z=0, & i.e.\ (0,\omega,-1) \quad \ldots \quad \ldots \quad (2) \\ y+\omega^2 z=0, & i.e.\ (0,\omega^2,-1) \quad \ldots \quad \ldots \quad (3) \end{cases}$$

$$y=0,\ \begin{cases} z+x=0, & i.e.\ (-1,0,1) \quad \ldots \quad \ldots \quad (4) \\ z+\omega x=0, & i.e.\ (-1,0,\omega) \quad \ldots \quad \ldots \quad (5) \\ z+\omega^2 x=0, & i.e.\ (-1,0,\omega^2) \quad \ldots \quad \ldots \quad (6) \end{cases}$$

$$z=0,\ \begin{cases} x+y=0, & i.e.\ (1,-1,0) \quad \ldots \quad \ldots \quad (7) \\ x+\omega y=0, & i.e.\ (\omega,-1,0) \quad \ldots \quad \ldots \quad (8) \\ x+\omega^2 y=0, & i.e.\ (\omega^2,-1,0) \quad \ldots \quad \ldots \quad (9) \end{cases}$$

It appears, therefore, that of these nine points of inflexion only *three* are real, and each of these lies on a side of the triangle of reference. These three real points are 1, 4 and 7; and they lie on a line whose equation is $x+y+z=0$.

185. The equation of the curve can be written in the form—

$$f \equiv x^3+y^3+z^3-3xyz+3(2m+1)xyz=0, \qquad \ldots \quad (1)$$

and consequently in the three following forms :—

$$\left.\begin{array}{l} (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)+3(2m+1)xyz=0 \\ (\omega x+y+z)(x+y+\omega z)(\omega^2 x+y+\omega^2 z)+3(2m+\omega)xyz=0 \\ (\omega^2 x+y+z)(x+\omega^2 y+z)(\omega x+\omega y+z)+3(2m+\omega^2)xyz=0 \end{array}\right\} \quad (2)$$

These may again be written in the following abridged forms :—

$$\left.\begin{array}{l} a_1 a_2 a_3+\lambda xyz=0 \\ b_1 b_2 b_3+\mu xyz=0 \\ c_1 c_2 c_3+\nu xyz=0 \end{array}\right\} \qquad \ldots \qquad \ldots \quad (3)$$

Hence we see that the nine points of inflexion lie on the twelve lines (i) x, y, z, (ii) a_1, a_2, a_3, (iii) b_1, b_2, b_3, (iv) c_1, c_2, c_3. The four groups of three lines each form four triangles which are called *inflexional triangles.* We may express the grouping of these nine points by the following scheme :—

$$\begin{array}{cccccc} & b_1 & b_2 & b_3 & & \\ & \backslash & \backslash & \backslash & & \\ c_1 \,— & 1 & 2 & 3 & — x & \\ c_2 \,— & 4 & 5 & 6 & — y & \\ c_3 \,— & 7 & 8 & 9 & — z & \\ & | & | & | & & \\ & a_1 & a_2 & a_3 & & \end{array} \qquad \ldots \qquad \ldots \quad (4)$$

The four inflexional triangles are each formed by the three lines which pass through

(I) the three points in each horizontal line,

(II) the three points in each vertical line,

(III) the three points which give a positive term in (4) considered as a determinant,

(IV) the three points which give a negative term in the same determinant.

For instance, on b_1 lie the three points 1, 5, 9; on b_2 lie 2, 6, 7; on b_3 lie 3, 4, 8, and so on.

186. We have seen that there are always three real points of inflexion on a cubic. These three points are situated in a real right line, for the real line which joins two of them intersects the curve in a third real point, and this point is a point of inflexion. The two vertices of the triangle situated on this inflexional line are necessarily conjugate imaginaries, and the two lines passing through these vertices form with the real line an inflexional triangle $(a_1 a_2 a_3)$. Hence follows that all the other points of inflexion are imaginary, and in the triangle considered, the points of inflexion on one of the sides are conjugate to those on the other. If these conjugate points are joined, two by two, we obtain a second triangle (xyz) of which the sides are all real, and on such a side is situated a real point of inflexion. The two other triangles are completely imaginary and one of them is conjugate to the other; for otherwise, there will be more than three real points of inflexion, which is impossible. Hence there are three real points of inflexion, four real inflexional lines, one real inflexional triangle; and four vertices of these triangles are real. These latter are the vertices of the real triangle and the vertex opposite

to the line joining the three real points of inflexion in the triangle of which this line is a side.

187. *Given the equation of a cubic, referred to one of the four inflexional triangles, to obtain its equation referred to any other.*

Let $f \equiv x^3+y^3+z^3+6mxyz=o$ be the equation of the cubic, referred to the lines $x=o$, $y=o$, $z=o$. We shall obtain its equation with reference to the triangle whose sides are

$$X \equiv x+y+z=o,$$

$$Y \equiv x+\omega y+\omega^2 z=o,$$

$$Z \equiv x+\omega^2 y+\omega z=o.$$

Let $\phi=x^3+y^3+z^3$, and, $\psi=xyz$ and $\phi'=X^3+Y^3+Z^3$ and $\psi'=XYZ$.

Now, $\phi'=X^3+Y^3+Z^3=3(x^3+y^3+z^3)+18xyz=3\phi+18\psi$.

$$\psi'=XYZ=x^3+y^3+z^3+3(\omega+\omega^2)xyz=\phi-3\psi.$$

$$\therefore\ 9\phi=\phi'+6\psi' \text{ and } 27\psi=\phi'-3\psi'.$$

But the equation of the curve is $\phi+6m\psi=o$, or, $9\phi+54m\psi=o$, which gives $\phi'+6\psi'+2m(\phi'-3\psi')=o$,

or, $$(1+2m)\phi'+6(1-m)\psi'=o,$$

i.e., $(1+2m)(X^3+Y^3+Z^3)+6(1-m)XYZ=o$.

188. Equations of the harmonic polars :

The co-ordinates of a point of inflexion on the side $x=o$ are $(0, 1, -1)$; the polar conic of this point is therefore $(y^2+2mxz)-(z^2+2mxy)=o$, *i.e.*, $(y-z)(y+z-2mx)=o$. Therefore, $y-z=o$ is the harmonic polar and $y+z-2mx=o$ is the corresponding inflexional tangent. Similarly, the harmonic polars of the other two points of inflexion on the side $x=o$ are $y-\omega z=o$ and $y-\omega^2 z=o$.

Thus the nine harmonic polars are—

$$\left.\begin{array}{l} y-z=0 \\ y-\omega z=0 \\ y-\omega^2 z=0 \end{array}\right\} \qquad \left.\begin{array}{l} z-x=0 \\ z-\omega x=0 \\ z-\omega^2 x=0 \end{array}\right\} \qquad \left.\begin{array}{l} x-y=0 \\ x-\omega y=0 \\ x-\omega^2 y=0 \end{array}\right\}$$

From this it follows at once that the harmonic polars of three collinear points of inflexion meet in a point.

189. Hesse's Theorem :

All cubics described through the nine points of inflexion on a non-singular cubic will have these points for points of inflexion.

Let $f \equiv x^3+y^3+z^3+6mxyz=0$ be the equation of a cubic. ... ... (1)

The equation of the Hessian is—

$$H \equiv m^2(x^3+y^3+z^3)-(1+2m^3)xyz=0. \qquad \dots \quad (2)$$

NOTE :—The form of this equation shows that it belongs to the same system of cubics as are given by (1), where m is considered as a parameter. Now the equation of the Hessian involves m in the third degree. Hence if we are given a cubic $H=0$, there are three different curves of the system (1) for which $H=0$ is the Hessian.

Now the points of inflexion on a curve are its intersections with the Hessian and therefore the equation of a curve passing through these points of inflexion is of the form $f+\lambda H=0$, where λ is a parameter, *i.e.*, the equation of a cubic passing through the points of inflexion on the cubic is $f+\lambda H \equiv (1+\lambda m^2)(x^3+y^3+z^3)+(6m-\lambda-2\lambda m^3)xyz=0$, which is also of the canonical form $x^3+y^3+z^3+6kxyz=0$,

where $6k \equiv \dfrac{6m-\lambda-2\lambda m^3}{1+\lambda m^2}$.

Therefore the points where it meets the sides of the triangle of reference are its points of inflexion. Hence

the points of inflexion on a cubic are also the points of inflexion on any cubic passing through them.

190. The canonical form of a nodal cubic :

The equation of every nodal cubic can be reduced to the form $x^3+y^3+6mxyz=o$, (1). That this transformation is possible is evident from the fact that the equation contains $3\times2+1=7$ free constants, which is the number of coefficients in the general equation of a cubic having a node at the origin. Consider a line $y=\lambda x$ passing through C, the third vertex of the triangle of reference. The points of intersection with the cubic are given by $x^3(1+\lambda^3)+6m\lambda x^2z=o$.

This equation, regarded as a cubic in $\frac{x}{z}$ is—

$(1+\lambda^3)\frac{x^3}{z^3}+6m\lambda\frac{x^2}{z^2}=o$. Hence two roots of this equation are zero, and therefore two of the points of intersection coincide with C. Consequently C is a double point on the curve, and $x=o$ is the tangent. Similarly $y=o$ is a tangent at C. Thus the point C is a node with the nodal tangents $x=o$, $y=o$.

Again, the equation can be written as $x^3+y^3+(-2mz)^3-3xy(-2mz)=(-2mz)^3$, which can again be put into the form $P.Q.R=z^3$, which shows that the points where $z=o$ meets the curve are points of inflexion and $P=o$, $Q=o$, $R=o$ are the tangents at these points. From Plücker's equations we see that a nodal cubic has only three points of inflexion and these lie on the line $z=o$.

Thus, to reduce the equation of a nodal cubic to the canonical form, the two nodal tangents and the line joining the three points of inflexion are to be taken as the sides of the triangle of reference.

191. The general equation of a cubic having the point C for a node is $u_3+u_2z=o$, where $u_2=o$ are the

nodal tangents. Therefore, if $x=o$ and $y=o$ are the nodal tangents, the equation can be written as—

$$ax^3+3bx^2y+3cxy^2+dy^3+kxyz=o. \quad \ldots \quad (1)$$

Now, the equation of the Hessian of this cubic is

$$k\,yz-3(ax^3 \mp bx^2y-cxy^2+dy^3)=o. \quad \ldots \quad (2)$$

The points of inflexion are the points of intersection of (1) and (2); and if $z=o$ is to pass through these intersections, it must intersect them both in the same points *i.e.*, the two equations, obtained by putting $z=o$ in (1) and (2),

namely, $\qquad ax^3+3bx^2y+3cxy^2+dy^3=o,$

and $\qquad ax^3-bx^2y-cxy^2+dy^3=o$

are to be the same, which requires that either $a=d=o$; or, $b=c=o$. But when $a=d=o$, the cubic reduces to three right lines.

Therefore we must have $b=c=o$, and the cubic reduces to $ax^3+dy^3+kxyz=o$.

Now, put $X=a^{\frac{1}{3}}x$, $Y=d^{\frac{1}{3}}y$ and $Z=z$, and the equation becomes

$$X^3+Y^3+6mXYZ=o, \text{ where } 6m=\frac{k}{a^{\frac{1}{3}}d^{\frac{1}{3}}}.$$

192. The equation of a nodal cubic can also be reduced to the canonical form $(y^2-kx^2)z-x^3=o$.

For, the equation of a cubic having a node at C can be written as $u_2z+u_3=o$, or $(y+mx)(y+m'x)z$

$$+(ax^3+3bx^2y+3cxy^2+dy^3)=o,$$

where $y+mx$ and $y+m'x$ are the nodal tangents. Next suppose that the vertex B is a point of inflexion, with $z=o$ as the inflexional tangent. Then the co-efficient of

$y^3=0$, and those of y^2 and y should have the common factor z. This requires that $d=0$, and $c=b=0$.

Thus the equation reduces to

$$(y+mx)(y+m'x)z+ax^3=0.$$

If further we take for $y=0$ the line which is the harmonic conjugate of $x=0$ with respect to the nodal tangents, we must have $m=-m'$ and the equation becomes

$$(y+mx)(y-mx)z+ax^3=0,$$

which may be written in the form

$$(y^2-kx^2)z-x^3=0. \qquad \ldots \qquad (1)$$

This is a second canonical form of a nodal cubic.

The equation (1) will represent a cubic having a node, a cusp, or a conjugate point at C, according as $k \gtreqless 0$.*

193. The canonical form of a cuspidal cubic.

In the case of a cuspidal cubic, there is only one point of inflexion. Let the vertex A be the point of inflexion with $y=0$ as the inflexional tangent. Then the equation of the cubic may be written as—

$$ax^2y+xyv_1+u_3=0. \ldots \qquad \ldots \qquad (1)$$

If the vertex B is a cusp, the equation should not contain y^3 and y^2; and if $x=0$ is the cuspidal tangent, the coefficient of y should be x^2 and the equation reduces to the form

$$ax^2y+bz^3=0.$$

* See H. Durège—"Über fortgesetztes Tantentenziehen an Kurven 3. Ordnung mit einem Doppel or Rückkehrpunkte." Math. Ann. Bd. I, 1869, pp. 509-532.

Now, put $X=x$, $Y=ay$ and $Z=-b^{\frac{1}{3}}z$ and the equation becomes

$$X^2 Y=Z^3. \quad \ldots \quad \ldots \quad (2)$$

Thus we see that every cuspidal cubic can be reduced to the form $x^2y=z^3$, the cuspidal tangent and the inflexional tangent being taken as two sides and the line joining the cusp to the point of inflexion as the third side of the triangle of reference.

Note : The Hessian of a cuspidal cubic is composed of three right lines, of which two coincide with the cuspidal tangent. For, the cuspidal cubic is $x^2y=z^3$, where $x=o$ is the cuspidal tangent. The Hessian of this cubic is $x^2z=o$, which proves the proposition.

194. *To find the condition, both necessary and sufficient, that three points on a cubic should be collinear.**

Let the equation of the cubic be given by

$$x^3+y^3+z^3+6mxyz=o, \ldots \quad \ldots \quad (1)$$

and that of a line by $\lambda x+\mu y+\nu z=o$. ... ... (2)

To find the points of intersection of (1) and (2), we eliminate z between them and the resulting equation is

$$x^3+y^3+\left(-\frac{\lambda x+\mu y}{\nu}\right)^3+6mxy\left(-\frac{\lambda x+\mu y}{\nu}\right)=o;$$

$$\text{or } (\nu^3-\lambda^3)\frac{x^3}{y^3}-3(\lambda^2\mu+2m\lambda\nu^2)\frac{x^2}{y^2}-3(\lambda\mu^2+2m\mu\nu^2)\frac{x}{y}$$
$$+(\nu^3-\mu^3)=o. \quad \ldots \quad (3)$$

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* These theorems on the intersections of a cubic and a line are taken from Cayley—" A memoir on curves of the third order "—Phil. Trans. of the R. Soc. of London Vol. CXLVII (1857).

Coll. Papers—Vol. II, No. 146, §§ 29—31.

Let (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_5, z_3) be the co-ordinates of the three points of intersection of (2) with (1).

Then from the equation (3) we obtain

$$\frac{x_1 x_2 x_3}{y_1 y_2 y_3} = -\frac{\nu^3 - \mu^3}{\nu^3 - \lambda^3} = \frac{\mu^3 - \nu^3}{\nu^3 - \lambda^3} \quad \ldots \quad \ldots \quad (4)$$

Similarly, by eliminating x between (1) and (2) we obtain

$$\frac{z_1 z_2 z_3}{y_1 y_2 y_3} = -\frac{\lambda^3 - \mu^3}{\lambda^3 - \nu^3} = \frac{\lambda^3 - \mu^3}{\nu^3 - \lambda^3} \quad \ldots \quad \ldots \quad (5)$$

From (4) and (5) we have

$$\frac{x_1 x_2 x_3}{\mu^3 - \nu^3} = \frac{y_1 y_2 y_3}{\nu^3 - \lambda^3} = \frac{z_1 z_2 z_3}{\lambda^3 - \mu^3} = \frac{x_1 x_2 x_3 + y_1 y_2 y_3 + z_1 z_2 z_3}{o} \quad \ldots \quad (5')$$

$$\therefore \; x_1 x_2 x_3 + y_1 y_2 y_3 + z_1 z_2 z_3 = o \quad \ldots \quad (6)$$

is the necessary condition that three points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) on a cubic should lie on a right line.

To prove that the condition (6) is sufficient :—

Let a line $\lambda x + \mu y + \nu z = o$ pass through two of the points, namely, (x_1, y_1, z_1) and (x_2, y_2, z_2) and intersect the cubic in a third point (x_3', y_3', z_3')

Then, as before, $x_1 x_2 x_3' + y_1 y_2 y_3' + z_1 z_2 z_3' = o.$

But $\qquad x_1 x_2 x_3 + y_1 y_2 y_3 + z_1 z_2 z_3 = o.$

$\therefore$ Either $x_3 = x_3'$, $y_3 = y_3'$, $z_3 = z_3'$; or each of $x_3', y_3', z_3' = o$, which is absurd. Therefore the condition (6) is sufficient, it is also necessary.

195. Let any two of the three points A (x_1, y_1, z_1), B(x_2, y_2, z_2), C(x_3, y_3, z_3) coincide, namely A and B. Then we have

$$x_1^2 x_3 + y_1^2 y_3 + z_1^2 z_3 = o. \quad \ldots \quad \ldots \quad (1)$$

Now, if the equation of the tangent to the cubic at A is

$$\lambda x+\mu y+\nu z=o,$$

then by (5′), $x_1{}^2x_3 : y_1{}^2y_3 : z_1{}^2z_3=\mu^3-\nu^3 : \nu^3-\lambda^3 : \lambda^3-\mu^3$;

and also $\lambda : \mu : \nu=(x_1{}^2+2my_1z_1) : (y_1{}^2+2mz_1x_1) :$
$(z_1{}^2+2mx_1y_1)$.

Thus we have $\mu^3-\nu^3=k(y_1{}^3-z_1{}^3)(y_1{}^3+z_1{}^3$
$+6mx_1y_1z_1-8m^3x_1{}^3)$

$$=-k(1+8m^3)(y_1{}^3-z_1{}^3)x_1{}^3,$$

since (x_1,y_1,z_1) is a point on the cubic. Similarly, finding the values of $\nu^3-\lambda^3$ and $\lambda^3-\mu^3$, we obtain finally

$$x_1{}^2x_3 : y_1{}^2y_3 : z_1{}^2z_3=\mu^3-\nu^3 : \nu^3-\lambda^3 : \lambda^3-\mu^3$$
$$=x_1{}^3(y_1{}^3-z_1{}^3) : y_1{}^3(z_1{}^3-x_1{}^3) : z_1{}^3(x_1{}^3-y_1{}^3).$$

Therefore $x_3 : y_3 : z_3=x_1(y_1{}^3-z_1{}^3) : y_1(z_1{}^3-x_1{}^3) :$
$z_1(x_1{}^3-y_1{}^3)$.

i.e., the co-ordinates of the tangential point are proportional to

$$x_1(y_1{}^3-z_1{}^3) : y_1(z_1{}^3-x_1{}^3) : z_1(x_1{}^3-y_1{}^3).$$

If all the three points A, B, C coincide, $x_1{}^3+y_1{}^3+z_1{}^3=0$, and consequently $x_1y_1z_1=0$. Therefore at a point of inflexion we must have

$$\left.\begin{aligned} x_1{}^3+y_1{}^3+z_1{}^3=0, \\ x_1y_1z_1=0. \end{aligned}\right\}$$

196. Salmon's Theorem :*

*The anharmonic ratio of the pencil formed by the four tangents which can be drawn from any point of a cubic is constant.**

* Salmon—Higher Plane curves, §. 167. For an algebraic proof of the theorem, see Journal für de Math. Bd. 42, 1851, p. 274.

We have seen that the polar conic of a point O on the cubic passes through the points of contact A,B,C,D of the tangents drawn from O and that it touches the cubic at the point O, *i.e.*, the polar conic of O passes through a consecutive point O′ and the points of contact A,B,C,D. Again the four tangents from O′ are infinitely near to OA, OB, OC, OD, and intersect them at their points of contact A,B,C,D. Thus the six points O, O′, A,B,C,D lie on a conic and therefore O(ABCD)=O′(ABCD). Hence we see that this ratio remains the same as we pass from any point of the curve to the consecutive one. Hence it is constant for all points of the cubic.

197. From what has been said above it is evident that each cubic is characterised by a definite value of this anharmonic ratio, which is an absolute invariant for any projective transformation, and this is the single absolute invariant which the curve possesses.*

We shall now determine the relation between this anharmonic ratio σ and the constant m, which enters the canonical form of the cubic, and is regarded as a parameter giving different curves of the system. For each particular value of m, we do not obtain a definite value of σ, because as we know, the anharmonic ratio has six different values† according to the different orders in which the four tangents are taken. These six values are

$$\sigma,\ 1/\sigma,\ 1-\sigma,\ 1/(1-\sigma),\ 1-\frac{1}{\sigma},\ 1/(1-\frac{1}{\sigma})$$

and are interchangeable in any expression. We have accordingly to form an equation which is satisfied by

* J. Thomæ—Über orthogonale Invarianten der Kurven dritter Ordnung, (Leipziger, Ber. 51, 1899)

† C. A. Scott—loc. cit. §37,

these six values. The co-efficient of this equation must be symmetric functions of the co-efficients in the equation of the four tangents.

If the equation of the four tangents be given in the form

$$a_0x^4+4a_1x^3y+6a_2x^2y^2+4a_3xy^3+a_4y^4=o, \ \ldots \quad (1)$$

then the equation for the six anharmonic ratios is

$$I^3\left\{(\sigma+1)(\sigma-2)(\sigma-\tfrac{1}{2})\right\}^2=27J^2\left\{(\sigma+w)(\sigma+w^2)\right\}^3 * (2)$$

where $I\equiv a_0a_4-4a_1a_3+3a_2^2$ and

$$J\equiv a_0a_2a_4+2a_1a_2a_3-a_0a_3^2-a_4a_1^2-a_2^3,$$

and ω, ω^2 are the imaginary cube roots of unity.

This equation is of the sixth degree in σ.

Now, the equation of the four tangents drawn from any point of the cubic $f\equiv x^3+y^3+z^3+6mxyz=0$ is obtained by applying the formula (3) of §. 51. In this case $f'=o$, since the point (x',y',z') lies on the curve. Thus the equation of the four tangents becomes

$$(\Delta'f')^2[(\Delta f)^2-4f.\Delta'f']=o$$

or $\quad P\equiv(\Delta f)^2-4f.\Delta'f'=o$, where $\Delta\equiv x'\dfrac{d}{dx}+y'\dfrac{d}{dy}+z'\dfrac{d}{dz}$,

$$\Delta'\equiv x\frac{d}{dx'}+y\frac{d}{dy'}+z\frac{d}{dz'}.$$

Since the ratio is the same for all points on the curve, we may consider the tangents drawn from the point (0, 1,—1).

* Burnside and Panton—Theory of Equations, Vol. I, *Ex*, 16, pp, 148—150,

Then, $\Delta f = \left(\frac{d}{dy} - \frac{d}{dz}\right) (x^3 + y^3 + z^3 + 6mxyz)$

$$=3(y^2 - z^2 + 2mzx - 2mxy)$$

$$=3(y-z)(y+z-2mx)$$

and $\Delta' f' = \left(x\frac{d}{dx'} + y\frac{d}{dy'} + z\frac{d}{dz'}\right) (x'^3 + y'^3 + z'^3 + 6mx'y'z')$

$$=x(3x'^2 + 6my'z') + y(3y'^2 + 6mz'x') + z(3z'^2 + 6mx'y')$$

$$=x(-6m) + y(3) + z(3)$$

$$=3(y+z-2mx).$$

$\therefore \quad P \equiv \left\{ 3(y-z)(y+z-2mx) \right\}^2 - 4.3(y+z-2mx) \times f$

$$=3(y+z-2mx)\left\{3(y-z)^2(y+z-2mx) - 4(x^3+y^3+z^3+6mxyz)\right\} = o.$$

Consider the section of this pencil by the line $z=o$.

Then, $(2mx-y)(4x^3+y^3+6mxy^2)=o$

or $8mx^4 - 4x^3y + 12m^2x^2y^2 - 4mxy^3 - y^4 = o$ ... (3)

Comparing the equations (1) and (3) we obtain

$a_0 = 8m$; $a_1 = -1$; $a_2 = 2m^2$; $a_3 = -m$; $a_4 = -1$.

$\therefore \quad I = 8m(-1) - 4(-1)(-m) + 3(2m^2)^2$

$$=12m(m^3-1)$$

$J = 8m.\ 2m^2.\ (-1) + 2(-1)(2m^2)(-m) - 8m(-m)^2 - (-1)(-1)^2 - (2m^2)^3$

$$=-8m^6 - 20m^3 + 1$$

Then the equation (2) becomes

$$\frac{16m^3(m^3-1)^3}{(8m^6+20m^3-1)^2}=\frac{(\sigma^2-\sigma+1)^3}{(\sigma+1)^2(\sigma-2)^2(2\sigma-1)^2} \quad \cdots \quad (4)$$

This determines twelve different cubics corresponding to any given value of σ.

198. If σ is a root of $\sigma^2-\sigma+1=o$, there are only four different values of m and the curves coincide three at a time; *i.e.*, when σ is equal to $-\omega$ or $-\omega^2$, where ω and ω^2 are the imaginary cube roots of unity, the ratios become equal three by three, and are said to be *equianharmonic*.

If σ is a root of any one of the equations $\sigma+1=o$, $\sigma-2=o$, $2\sigma-1=o$, there are only six different values of m and the curves coincide two at a time; *i.e.*, if $\sigma=-1$, or $1/\sigma=-1$, then the remaining four values become respectively equal to 2 and $\frac{1}{2}$, two by two. This ratio is said to be *harmonic*.

The curves corresponding to these different ratios have their corresponding names, and we say that in the system of cubics $x^3+y^3+z^3+6mxyz=o$, corresponding to each value of σ *there are twelve different cubics, and there are four harmonic and six ~~equi~~anharmonic curves in the system.*

For all equianharmonic curves, $\sigma^2-\sigma+1=o$, and consequently $I=o$, and this gives two real values of m.

For all harmonic curves, $J=o$ and there are only two real values of m.

If $\sigma=1$, 0 or ∞, then the six ratios coincide in pairs and they are respectively equal to $1, 0, \infty$. If the curve has a double point, two of the four tangents must coincide and the discriminant of the equation (3) must vanish. This requires that $I^3-27J^2=o$. The expression for this in terms of m is $(1+8m^3)^3=o$.

Therefore the condition for a double point is $1+8m^3=o$.

We obtain the degenerate cubic consisting of the sides of the fundamental triangle when $m=\infty$.

Harmonic cubics also possess the property that the Hessian of the Hessian coincides with the curve. But this requires that

$$-6m=\frac{1+2m'^3}{m'^2} \text{ and } -6m'=\frac{1+2m^3}{m^2}.$$

where $H\equiv x^3+y^3+z^3+6m'xyz=o$.

From this we obtain $-6m=\dfrac{1+2\left(-\dfrac{1+2m^3}{6m^2}\right)^3}{\left(-\dfrac{1+2m^3}{6m^2}\right)^2}$

or, $1+2\left(-\dfrac{1+2m^3}{6m^2}\right)^3+6m\left(-\dfrac{1+2m^3}{6m^2}\right)^2=o$,

or, $64m^9+168m^6+12m^3-1=(8m^6+20m^3-1)(8m^3+1)$

$$=-J(8m^3+1)=o,$$

which is satisfied, since the cubics are harmonic.

199. *If the polar conic of a point A with respect to a cubic breaks up into two right lines intersecting at B, the polar conic of B breaks up into two right lines intersecting at A.*

This is a particular case of the theorem of § 71, for the first polar of a cubic is its polar conic.

Let $f\equiv x^3+y^3+z^3+6mxyz=o$ be the given cubic and (x_1,y_1,z_1) and (x_2,y_2,z_2) be the co-ordinates of A and B respectively.

The polar conic of A is $\phi\equiv x_1(x^2+2myz)+y_1(y^2+2mzx)$

$$+z_1(z^2+2mxy)=o.$$

If this has a double point at $B(x_2,y_2,z_2)$, we must have

$$\frac{d\phi}{dx_2}=o, \quad \frac{d\phi}{dy_2}=o, \quad \frac{d\phi}{dz_2}=o;$$

$$\left.\begin{aligned} x_1x_2+m(y_1z_2+y_2z_1)&=o \\ y_1y_2+m(z_1x_2+z_2x_1)&=o \\ z_1z_2+m(x_1y_2+x_2y_1)&=o \end{aligned}\right\} \quad \ldots \quad (A)$$

i.e.

Now, since these equations are symmetrical, in the co-ordinates of the two points, it follows that the polar conic of B also breaks up into two right lines intersecting at A. The locus of both these points is therefore the Hessian, and it follows in consequence that the Steinerian of a cubic coincides with its Hessian.

By eliminating (x_2, y_2, z_2) between the equations (A), we find

$$\begin{vmatrix} x_1 & mz_1 & my_1 \\ mz_1 & y_1 & mx_1 \\ my_1 & mx_1 & z_1 \end{vmatrix}=0$$

which, on simplification, becomes

$$m^2(x_1{}^3+y_1{}^3+z_1{}^3)-(1+2m^3)x_1y_1z_1=o.$$

This shows that the locus of (x_1, y_1, z_1) is the Hessian. Similarly, by eliminating (x_1, y_1, z_1) it may be shown that (x_2, y_2, z_2) lies on the Hessian.

These two points A and B are called by Professor Cayley *conjugate poles*.*

200. Net of Polar Conics :

The equation of the polar conic of a point (x', y', z') is

$$x'\frac{df}{dx}+y'\frac{df}{dy}+z'\frac{df}{dz}=o \quad \ldots \quad \ldots \quad (1)$$

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* Cayley—" A memoir on curves of the third order," Coll. Papers, Vol. II, p. 382.

Thus the system of polar conics of a cubic curve forms a *net*, if we consider (x', y', z') as parameters. The equation (1) represents the doubly infinite system of first polars, which for the cubic forms a net. For, in this case, the polar system is determined by 15 constants, 9 in the equation of the curve and $3 \times 2 = 6$ co-ordinates of the vertices of the triangle of reference, while a net of conics contains $3 \times 5 = 15$ constants.

Note: In general, a net of curves of order m may be considered as a system of first polars of a curve of order $(m+1)$. Any three curves of the net generally involves $\frac{3}{2}m(m+3)$ constants. The first polars of three different points involve $\frac{(m+1)(m+4)}{2}$ constants of the original curve and the six co-ordinates of the three points and therefore in all $\frac{(m+1)(m+4)}{2}+6$ constants. These two numbers will be equal if $m=2$. Hence a net of conics can always be considered as the system of first polars of a curve of the third order.

It should be remembered that so long as we deal with a pencil of conics, each point A corresponds to the conjugate point B, through which pass the polars of A with respect to the same pencil. But in the case of a net, this is not in general the case. There is a series of points A whose polars with respect to three, and therefore all, conics of the net pass through one and the same point B. The locus of A is called the " Jacobian " of the net of conics. The points B also lie on this curve.

The polar line, with respect to $f_x \equiv x^2 + 2myz = o$, of any point $A(x_1, y_1, z_1)$ is $xx_1 + m(yz_1 + y_1 z) = o$.

If this passes through B (x_2, y_2, z_2), we have

$$x_1 x_2 + m(z_1 y_2 + y_1 z_2) = o \qquad \ldots \qquad \ldots \qquad (1)$$

Similarly, if the polars of A with respect to the two conics $f_y \equiv y^2 + 2mzx = o$ and $f_z \equiv z^2 + 2mxy = o$ pass

through B, we must have

$$y_1 y_2 + m(z_1 x_2 + z_2 x_1) = o \quad \dots \quad (2)$$

$$z_1 z_2 + m(x_1 y_2 + x_2 y_1) = o \quad \dots \quad (3)$$

Thus we see that in this case A and B are conjugate points with respect to each of the conics $f_x = o, f_y = o, f_z = o$; and therefore A and B are points on the Jacobian of the three conics.* But when the polars of a point with regard to three conics u, v, w are concurrent, the polars with respect to all conics of the system $lu + mv + nw = o$ will pass through the same point. Hence the Hessian of of a cubic is the Jacobian of the net of polar conics $lf_x + mf_y + nf_z = o$. The line AB is cut in involution by the system of polar conics, of which A and B are the foci. If $lf_x + mf^y + nf_z = o$ breaks up into two right lines, they intersect on AB and touch the Cayleyan.

201. *The polar line with respect to the cubic of a point A on the Hessian touches this latter curve at the conjugate pole* B.

The equation of the cubic being $f = x^3 + y^3 + z^3 + 6mxyz = 0$, that of the polar line of a point A (x_1, y_1, z_1) is

$$x(x_1{}^2 + 2\,my_1\,z_1) + y(y_1{}^2 + 2\,mz_1 x_1)$$
$$+ z(z_1{}^2 + 2mx_1 y_1) = 0 \quad \dots \quad (1)$$

The relations between two conjugate poles A and B are given by the equations (A) of §. 199. From the first and third of these equations we obtain

$$\frac{x_1}{mz_2^2 - m^2 x_2 y_2} = \frac{y_1}{m^2 y_2^2 - z_2 x_2} = \frac{z_1}{mx_2^2 - m^2 y_2 z_2} = k \text{ (say)}$$

Then, $x_1{}^2 + 2my_1 z_1 = k^2 [(mz_2^2 - m^2 x_2 y_2)^2 + 2m(m^2 y_2^2 - z_2 x_2) \times (mx_2^2 - m^2 y_2 z_2)]$

$$= k^2 [3m^4 x_2^2 y_2^2 + m^2 z_2 (z_2{}^3 - 2m^3 y_2^3 - 2x_2^3)]$$

* Salmon—Conic Sections, §. 388.

But (x_2, y_2, z_2) being a point on the Hessian,

$$m^2(x_2^3+y_2^3+z_2^3)-(1+2m^3)x_2y_2z_2=0.$$

$$\therefore \quad m^2z_2{}^3=(1+2m^3)x_2y_2z_2-m^2(x_2{}^3+y_2{}^3);$$

$$\therefore \quad x_1{}^2+2m_1y_1z_1=k^2\,[3m^4x_2{}^2y_2{}^2+(1+2m^3)$$
$$\times(x_2y_2z_2{}^2-m^2y_2{}^3z_2)-3m^2x_2{}^3z_2]$$
$$=k^2(m^2y_2{}^2-z_2x_2)\,[3m^2x_2{}^2-(1+2m^3)y_2z_2]$$

Similarly,

$$y_1{}^2+2mz_1x_1=k^2(m^2y_2{}^2-z_2x_2)\quad[3m^2y_2{}^2-(1+2m^3)z_2x_2]$$

and $z_1{}^2+2mx_1y_1=k^2(m^2y_2-z_2x_2)\,[3m^2z_2{}^2-(1+2m^3)x_2y_2]$

$\therefore$ The equation of the tangent (1) becomes

$$x[3m^2x_2{}^2-(1+2m^3)y_2z_2]+y[3m^2y_2{}^2-(1+2m^3)z_2x_2]+$$
$$z[3m^2z_2{}^2-(1+2m^3)x_2y_2]=0$$

which is the equation to the tangent to the Hessian at the point B (x_2, y_2, z_2).

A simple geometrical proof may also be given by considering the Hessian as the envelope of lines two of whose poles, with respect to the cubic, coincide, when the tangent at any point on the Hessian becomes the polar line of the conjugate point.

202.✓ *The tangents to the Hessian at two conjugate poles A, B intersect on the Hessian at a point C which is conjugate to the third point in which AB intersects the Hessian again.*

Let the two lines through B which constitute the polar conic of A intersect the corresponding lines through A in the four points P, Q, R, S. Then all conics passing through these four points have their poles on the line AB. The lines PR and QS may be taken to constitute the polar conic of some point on AB which must be a point on the Hessian. Therefore PR, QS constitute

the polar conic of the point C' where AB meets the Hessian again. If they intersect at C, then C is a point on the Hessian which is conjugate to C'. (Fig. 25.)

Again, the tangent to the Hessian at B is the polar line of A with respect to the cubic, which is also the polar line with respect to the polar conic of A *i.e.* w.r.t. the conic consisting of PS, QR. But the points A,B,C are the vertices of a triangle self-polar for all conics through P, Q, R, S. Hence BC is the polar of A and it is therefore the tangent to the Hessian at B. Similarly AC is the tangent at A. Hence the tangent at A and B intersect at C on the Hessian, which is the point conjugate to C'. Thus *two conjugate points have the same tangential point on the Hessian.*

203. In the preceding article, since the polar lines of A and B intersect at C, the first polar (polar conic) of C with respect to the cubic must pass through A and B. (§. 56), and as C is a point on the Hessian, its polar conic consists of a pair of lines. Hence the polar conic of C consists of the line AB and another line passing through C', the third point where AB meets the Hessian. Thus we obtain the following theorem :—

A line which joins two conjugate poles on the Hessian forms a part of the polar conic of the point conjugate to the third point, where the line intersects the Hessian.

204. If on the other hand, we are given the Hessian $H=0$, a curve of the third order, corresponding to any point A we have a conjugate point B which is the point of contact of a tangent drawn from C the tangential of A. But from C three other tangents, different from the tangent at A, can be drawn to the curve, whose points of contact B,B',B'' are conjugate points to A with respect to three different cubics, of which the given cubic $H=0$ may be

considered as the Hessian. The conjugate point-pairs therefore form three different systems.

If we start with any pair of conjugate points P and P′, with a common tangential point T, and join P,P′ to any point Q on the curve, then PQ and P′Q intersect the cubic again in the points R and R′ respectively. The lines PR and P′R′ intersect the curve at the same point Q′ conjugate to Q. For, let P_1 and P_1' be two other points on the curve consecutive to P and P′ respectively. Then $P_1PRP_1'P'R'$ is a hexagon inscribed in the cubic. Two pairs of opposite sides (P_1P, $P_1'P'$) and (PR, P′R′) intersect respectively in the points T and Q on the curve. The third pair $P_1'R$ and P_1R' (which ultimately coincide with P′R and PR′ respectively) intersect at the point Q′ on the curve. (§. 156).

Again RQR′Q′ is a quadrilateral inscribed in the cubic, whose opposite sides intersect at the points P and P′ on the curve. Therefore the opposite vertices Q, Q′ and R,R′ have the same tangential points, or in other words, Q and Q′ are conjugate poles, as also are R and R′. Hence we obtain the theorem :—

On each cubic there are three different systems of conjugate points such that any two pairs joined cross-wise give another pair of conjugate points.

The Cayleyan.

205. Definition : The envelope of the line joining two conjugate poles was called by Prof. Cayley the *Pippian* ; but it is called by Cremona the *Cayleyan.*

From §. 200 it follows that the Cayleyan is the envelope of lines which are cut in involution by the system of polar conics. It is in fact a contravariant of the cubic.

From what has been said in §. 200 it is evident that the Cayleyan is the envelope of the lines which constitute

the polar conics of points on the Hessian.* Hence we can deduce the tangential equation of the Cayleyan as follows :—

The equation of the polar conic of a point (x',y',z') with respect to a cubic $f=o$ is

$$a'x^2+b'y^2+c'z^2+2f'yz+2g'zx+2h'xy=o \quad \ldots \quad (2)$$

where $a,'b,'c,'$......are the second differential co-efficients of f, in which $x,'y,'z'$ have been substituted for x, y, z respectively.

If this breaks up into two lines, it must be identical

with $(\lambda x+\mu y+\nu z)\left(\dfrac{a'x}{\lambda}+\dfrac{b'y}{\mu}+\dfrac{c'z}{\nu}\right)=o \quad \ldots \quad (2)$

Comparing the co-efficients of yz, zx and xy in (1) and (2) we obtain

$$\left.\begin{aligned} 2f' &= \frac{b'\nu}{\mu}+\frac{c'\mu}{\nu} \\ 2g' &= \frac{c'\lambda}{\nu}+\frac{a'\nu}{\lambda} \\ 2h' &= \frac{a'\mu}{\lambda}+\frac{b'\lambda}{\mu}. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (3)$$

If we eliminate $(x,'y,'z')$ between equations (3), we obtain the tangential equation of the Cayleyan. The calculation becomes easier if the cubic be given in the canonical form $x^3+y^3+z^3+6mxyz=o$. Then, $a'=6x'$, $b'=6y'$ $c'=6z.'$ $f'=6mx,'$ $g'=6my,'$ $h=6mz'$ and the equations (3) become—

$$\left.\begin{aligned} x'\mu^2+y'\lambda^2-2m\lambda\mu z'&=o \\ x'\nu^2-2m\nu\lambda y'+\lambda^2 z'&=o \\ -2m\mu\nu x'+\nu^2 y'+\mu^2 z'&=o \end{aligned}\right\}$$

*In the case of curves of higher degrees the envelope of lines joining corresponding points is distinct from the envelope of lines into which the polar conics break up.

Eliminating (x',y',z') between these equations, we obtain

$$\begin{vmatrix} \mu^2 & \lambda^2 & -2m\lambda\mu \\ \nu^2 & -2m\lambda\nu & \lambda^2 \\ -2m\mu\nu & \nu^2 & \mu^2 \end{vmatrix}=o$$

which simplified gives the tangential equation of the Cayleyan in the canonical form :—

$$C\equiv m\ (\lambda^3+\mu^3+\nu^3)+(1-4m^3)\lambda\mu\nu=o \quad \ldots \quad (4)$$

omitting the factor $\lambda\mu\nu$, since it cannot form the part of the envelope. For, $\lambda=o$ represents simply a point. Similarly, $\mu=o$, $\nu=o$ are also points. But the Cayleyan is of the third class.

Note: If λ,μ,ν be eliminated between the equations (3), we obtain the equation of the Hessian. The form of this equation is the same as that of the given cubic, and therefore the Cayleyan C stands to the vertices of the fundamental triangle in the same relation as the original curve stands to the sides. The tangents to this curve drawn from the vertices correspond to the inflexional tangents on the original curve. If m is regarded as a parameter, the equation (4) represents the system of Cayleyan curves of the system of cubics defined by the canonical form.

206. *The Cayleyan of a cubic curve touches the nine harmonic polars, as well as the nine inflexional tangents.*

This immediately follows from the fact that the tangent to a cubic at a point of inflexion and the corresponding harmonic polar constitute the polar conic of the point of inflexion, and the Cayleyan being the envelope of degenerate polar conics must touch these lines.

This can also be shown directly by substituting the coordinates of the nine inflexional tangents and those of

the nine harmonic polars in the tangential equation of the Cayleyan. It will be found that the equation is satisfied by the coordinates of each of these lines.

207. *The nine harmonic polars of a cubic are cuspidal tangents to the Cayleyan.*

The tangential equation of the Cayleyan is

$$C \equiv m(\lambda^3+\mu^3+\nu^3)+(1-4m^3)\lambda\mu\nu=o \quad \dots \quad \dots \quad (1)$$

or, $\lambda^3+\mu^3+\nu^3+6k\lambda\mu\nu=o$, where $6k=\dfrac{1-4m^3}{m}$.

The equation can be written as

$$(\lambda+\mu-2k\nu)\ (\omega\lambda+\omega^2\mu-2k\nu)\ (\omega^2\lambda+\omega\mu-2k\nu)$$
$$=-(1+8k^3)\nu^3.$$

This equation is satisfied by

$$(2) \qquad \nu^3=o, \quad \begin{cases} \lambda+\mu-2k\nu=o\ ; \\ \omega\lambda+\omega^2\mu-2k\nu=o\ ; \\ \omega^2\lambda+\omega\mu-2k\nu=o\ ; \end{cases}$$

Let P be the point $\lambda+\mu-2k\nu=o$. Now, $\nu^3=o$ represents the third vertex C counted thrice. Therefore the lines joining P to three coincident points at C are tangents to the Cayleyan. Therefore, three contiguous tangents pass through P. Hence P is a cusp on the Cayleyan and PC is the cuspidal tangent. Similarly, the other two factors in (2) give two other cusps Q and R; and QC, RC are the cuspidal tangents. The line $\lambda+\mu-2k\nu=o$ and $\nu=o$ gives $\lambda+\mu=o$ *i.e.* $\mu=-\lambda$, and the point-equation of the line PC is $x-y=o$. Similarly, the point-equations of QC and RC are respectively $x-\omega y=o$, $x-\omega^2 y=o$; but these are the equations of the harmonic polars of the points of inflexion of the cubic on the line $z=o$ (§ 188). Similarly, it can be shown that all the other harmonic polars are cuspidal tangents to the Cayleyan.

Combining this with the theorem of the preceding article, since these equations are independent of m, we may deduce the following theorem:—

The nine inflexional tangents of a curve of the third order in a system having the same nine points of inflexion determine a curve of the third class, and all these curves of the third class have the same cuspidal tangents.

208. We have seen (§. 168) that the poles of a tangent to the Hessian at any point A are the two points of contact with the Cayleyan of the lines which constitute the polar conic of A and the point B′ counted twice, where B′ is the point conjugate to the tangential point B of A. Thus, the locus of the poles of the tangents to the Hessian includes the Cayleyan as well as the Hessian counted twice. Now the polar line of a point (x', y', z') with regard to the cubic $f \equiv x^3 + y^3 + z^3 + 6mxyz = o$ is

$$x \frac{df}{dx'} + y \frac{df}{dy'} + z \frac{df}{dz'} = o. \qquad \ldots \quad (1)$$

The condition that this should touch the Hessian involves the coefficients $\frac{df}{dx'}$, $\frac{df}{dy'}$, $\frac{df}{dz'}$ in the sixth degree, and therefore the variables (x', y', z') in the twelfth degree. Hence the locus of the poles is of twelfth degree, but contains the Hessian counted twice. Hence the Cayleyan is of degree six.

This can also be shown by forming the Cartesian equation of the Cayleyan from the tangential form we have already obtained. The Cartesian equation of the curve is obtained by eliminating ν between $\lambda x + \mu y + \nu z = o$ and the tangential equation of the curve, and then equating the discriminant of the resulting expression to zero. Thus

we obtain the equation of the Cayleyan in the form—

$$x^6+y^6+z^6-2(1+16k^3)(x^3y^3+y^3z^3+z^3x^3)$$
$$-24k^2xyz(x^3+y^3+z^3)-24k(1+2k^3)x^2y^2z^2=o,$$

where $6k\equiv\dfrac{1-4m^3}{m}$.

Note:—In the same way we obtain the tangential equation of the given cubic $x^3+y^3+z^3+6mxyz=o$ in the form—

$$\lambda^6+\mu^6+\nu^6-2(1+16m^3)(\lambda^3\mu^3+\mu^3\nu^3+\nu^3\lambda^3)$$
$$-24m^2\lambda\mu\nu(\lambda^3+\mu^3+\nu^3)-24m(1+2m^3)\lambda^2\mu^2\nu^2=o.$$

209. *The points of contact with the Cayleyan of the four lines which constitute the polar conics of two conjugate poles are collinear.*

Let AQ, AS and BQ, BS be the polar conics of two conjugate poles B and A respectively. (Fig. § 202). Let C′ be the point conjugate to C. The polar conic of C is the line AB and a line C′*abcd* drawn through C′. Let this line intersect AQ, AS, BS, BQ in the points *a*, *b*, *c*, *d* respectively. Then *a*, *b*, *c*, *d* will be the points of contact with the Cayleyan of those lines respectively.

Now the poles of AC are the points of intersection of the polar conics of A and C. But the polar conic of C consists of the lines AB and C′*abcd*, and that of A consists of the lines BQ, BS. Hence the poles are the points *c*, *d* and the point B counted twice. Therefore the points *c*, *d* are the points of contact of BS and BQ with the Cayleyan. Similarly, *a* and *b* are the points of contact of AQ and AS with the Cayleyan.

210. *If D be the point where AB touches the Cayleyan, then ABDC′ form a harmonic range.*

First, let us determine the point of contact of AB with the Cayleyan. From what has been said above, it is evident that we must first determine the complementary

point* C′ on AB and then join the two conjugate poles C and C′. Let D′ be the complementary point on CC′. Therefore the line CC′ and the conjugate line through D′ constitute the polar conic of some point on the Hessian. This conjugate line DD′ intersects AB in the required point of contact.

Now, every polar conic divides harmonically the line joining two conjugate poles (§. 200). Hence the lines CC′ and DD′ divide AB harmonically, which proves the theorem.

211. *The Hessian touches the nine inflexional tangents to a cubic at their intersections with the corresponding harmonic polars.*

We have shown that a point of inflexion A on a cubic is also a point of inflexion on the Hessian, but they have not the same inflexional tangent at A; for the tangent at A to the cubic is of the form $x+y-2mz=o$, which involves m and therefore cannot be the same for the curve and the Hessian. The polar conic of A consists of the tangent to the cubic at A and the corresponding harmonic polar intersecting the tangent at B. Then B is the point conjugate to A and both lie on the Hessian; and both the lines are tangents to the Cayleyan. The polar line of A with regard to the cubic is the tangent AB and therefore AB (§. 201) is the tangent to the Hessian at the conjugate pole B. Thus the inflexional tangent AB touches the Hessian at its intersection with the corresponding harmonic polar.

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* The point in which the line joining two conjugate poles meets the Hessian again is called by Salmon the *complementary point.* It is the point where the line is cut by the conjugate line, the two forming the polar conic of some point.—See H. P. curves, §. 176.

212. The line AB is an ordinary tangent to the Cayleyan, while the harmonic polar through B is a cuspidal tangent (§ 207). To find the point of contact with the Cayleyan of AB, we apply the method of § 209. Since AB is a tangent to the Hessian at B, the complementary point is B and the line through B, conjugate to AB, on which the point of contact must lie, is the harmonic polar, and it intersects AB at B, and therefore B is the point of contact of AB with the Cayleyan, *i.e.* the point of contact of the inflexional tangent with the Cayleyan is the point where it meets the harmonic polar. Therefore, the Hessian and the Cayleyan touch each other at B with the common tangent AB, which is the inflexional tangent to the cubic at the conjugate pole A. Thus we obtain the theorem :—

The Cayleyan and the Hessian touch each other at nine points, which are the nine points of intersection of the nine inflexional tangents with the corresponding harmonic polars, the common tangents at these points being the nine inflexional tangents to the cubic.

Note : These two curves cannot touch each other at more than nine points ; for the Cayleyan is of degree six and the Hessian of degree three. The number of common points is therefore 18 (§ 16). If they touch each other at more than nine points, they would intersect in more than 18 points which is impossible. The number of common tangents is also nine, for they can have only 18 common tangents, one being of class 6 and the other of class 3. When they touch at a point, the common tangent at that point counts as two common tangents.

213. The Pole Conic of a given line: *

Definition : The locus of points whose polar conics with regard to a given cubic touch a given line is a conic and is called the *Pole Conic* of the line.

Let $lx+my+nz=o$ be the equation of the given line and (x', y', z') the co-ordinates of a point. The polar

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* *Cf.* § 60.

conic of this point with regard to a cubic is

$$a'x^2+b'y^2+c'z^2+2f'yz+2g'zx+2h'xy=o,$$

where a', b', c', have the significance of § 205.

If this touch the given line $lx+my+nz=o$, the required condition is—

$$A'l^2+B'm^2+C'n^2+2F'mn+2G'nl+2H'lm=0\ldots \quad (1)$$

where A', B', C',...are the minors of a', b', c',...in the determinant

$$\begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix}$$

and are therefore functions of second degree in (x', y', z'). Hence the locus of (x', y', z') is a conic which is called the *Pole Conic* of the given line and its equation is $Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=o\ldots(2)$.

Note: If we consider (x', y', z') as fixed, the equation (1) represents tangentially the polar conic of the point.

The form of the equation (2) suggests another method of defining the Pole Conic as *the envelope of polar lines of points on the given line,* For, if (x', y', z') is a point on the line $lx+my+nz=o$, the polar line is $ax'^2+by'^2+cz'^2+2fy'z'+2gz'x'+2hx'y'=o\ldots\ldots(3)$ where a, b, c,... are the second differential coefficients of the function f. Then we are to find the envelope of (3), subject to the condition $lx'+my'+nz'=o$, which is the same thing as finding the condition that $lx+my+nz=o$ should touch the conic $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=o$ and it is therefore $Al^2+Bm^2+Cn^2+Fmn+2Gnl+2Hlm=o$.

214. *The pole conic of a line may also be defined as the locus of the poles of the line with regard to the polar conics of all the points on the line.*

The polar conic of any point (x', y', z') on the line $lx+my+nz=o$......(1) is $F\equiv x'f_x+y'f_y+z'f_z=o$......(2).

Let (x'', y'', z'') be the pole of the line with regard to (2). Therefore the equation (1) must be the same as $x''F_x+y''F_y+z''F_z=o$;

i.e. as $x(a''x'+h''y'+g''z')+y(h''x'+b''y'+f''z')$
$$+z(g''x'+f''y'+c''z')=o.$$

$$\therefore \quad \left.\begin{aligned} \lambda l &= a''x'+h''y'+g''z' \\ \lambda m &= h''x'+b''y'+f''z' \\ \lambda n &= g''x'+f''y'+c''z' \\ \text{and} \quad o &= lx'+my'+nz' \end{aligned}\right\}$$

Eliminating (x', y', z', λ) between these equations, we obtain as the locus of (x'', y'', z'') the equation (2) of the preceding article.

215. *The tangents to a curve are touched by their Pole Conics.*

We have shown that the pole conic of a line is touched by the polar lines of points on the given line. Now, if the line touches the cubic, the polar line of its point of contact is the line itself and this again touches the pole conic of this line ; *i.e.*, the tangent touches the pole conic with regard to a non-singular cubic.

This property has been used in finding the tangential equation of a cubic. The equation of the pole conic given in (2) of § 213 may be written as $A_1x^2+B_1y^2+C_1z^2+2F_1yz+2G_1zx+2H_1xy=o$, where A_1 B_1 C_1...are functions of the second degree in l, m, n. The condition that this may touch the line $lx+my+nz=o$ is $(B_1C_1-A_1^2)l^2+(C_1A_1-B_1^2)m^2+(A_1B_1-H_1^2)n^2+\ldots=o$, which is of the sixth degree in l, m, n and gives the tangential equation of the cubic.

Again, *if the line is a tangent to the Cayleyan, its Pole Conic reduces to a point.* Since the line is a tangent to the Cayleyan, this together with the conjugate line constitutes the polar conic of some point, and all the polar lines of points on the line pass through this point, and therefore the envelope reduces to the point.

216. The Pole Conic of the line at infinity.

When the given line lies at infinity, its pole conic is the locus of points whose polar conics touch the line at infinity, *i.e.*, become parabolas.*

The polar conic of a point (x', y', z') is $a'x^2+b'y^2+c'z^2+2f'yz+2g'zx+2h'xy=o$. This touches the line at infinity $\mathrm{a}x+\mathrm{b}y+\mathrm{c}z=o$ where, a, b, c are the sides of the triangle of reference, if

$$\begin{vmatrix} a' & h' & g' & \mathrm{a} \\ h' & b' & f' & \mathrm{b} \\ g' & f' & c' & \mathrm{c} \\ \mathrm{a} & \mathrm{b} & \mathrm{c} & 0 \end{vmatrix}=0. \quad \ldots \quad (1)$$

Therefore the locus of (x', y', z') is given by the determinant (1) after removing the dashes.

If the equation of the curve be given in Cartesian co-ordinates, the equation of the Pole Conic of the line at infinity becomes

$$f_{11}.f_{22}-(f_{12})^2=o; \; i.e., \frac{d^2f}{dx^2}\cdot\frac{d^2f}{dy^2}-\left(\frac{d^2f}{dxdy}\right)^2=o.$$

217.† From Maclaurin's theorem § 53, it follows that if a line intersect a cubic in the points P, Q, R, and tangents to the curve be drawn at these points, then the polar line of any point on the line with respect to the cubic is the same as its polar line with respect to the three

* See § 70.

† Salmon—H. P. Curves, § 185.

tangents. Now, if these tangents be $xyz=o$, the polar line of any point $(x', y', z',)$ is $xy'z'+yz'x'+zx'y'=o$... (1)

Also $$lx'+my'+nz'=o \quad \ldots \quad \ldots \quad (2)$$

Hence the pole conic of the line is the envelope of (1), subject to the condition (2). This is therefore

$$\sqrt{(lx)}+\sqrt{(my)}+\sqrt{(nz)}=o. \quad \ldots \quad (3)$$

The form of the equation shows that the conic (3) touches the three tangents, *i.e.*, the sides of the triangle of reference ABC at the points D,E,F (say) (Fig. 26). From each of P,Q,R a second tangent can be drawn to the conic. If the points of contact of these tangents be L,M,N respectively, then by the properties of the inscribed conic, the groups A,L,D ; B,M,E ; and C,N,F are each collinear and these lines meet in a point. By the harmonic properties of quadrilaterals, the points D,E,F are harmonic conjugates of P,Q,R with respect to the point-pairs B,C ; C,A ; and A,B respectively.

218. *The points of intersection of a line with its pole conic are situated equianharmonically with regard to the points where the line intersects the original curve.*

Let $f \equiv x^3+y^3+z^3+6mxyz=o$ be the given cubic. Let us consider the pole conic of $x=o$. The pole conic of $x=o$ is $yz=m^2x^2$, *i.e.* $y=o$ and $z=o$ are the tangents to the conic, where $x=o$ meets it. The line intersects the cubic where $y^3+z^3=o$. Thus the lines joining the vertex A with these points are $y+z=o$, $y+\omega z=o$, $y+\omega^2 z=o$....(1) These three lines together with the line $y=o$ form an equianharmonic pencil, for the six anharmonic ratios of this pencil are $-\omega, -\omega, -\omega; -\omega^2, -\omega^2, -\omega^2$. Similarly, $z=o$ forms with the lines (1) an equianharmonic pencil. Therefore the transversal $x=o$ meets both these pencils in equianharmonic ranges, which proves the proposition.

CHAPTER XIII.

UNICURSAL CUBICS.

219. In article 41 we have shown that a curve of deficiency zero is unicursal, *i.e.* if a curve have its maximum number of double points, the coordinates of any point on the curve can be expressed as rational and integral functions of a variable parameter. In the case of curves of third order, the maximum number of double points it can possess is *one*, and therefore the coordinates of any point on an autotomic cubic can be expressed as rational and integral functions of a variable parameter. Thus the equation of a unicursal cubic can be expressed in terms of a parameter as follows :—

$$\left.\begin{aligned} x&=a_0\lambda^3+a_1\lambda^2+a_2\lambda+a_3, \\ y&=b_0\lambda^3+b_1\lambda^2+b_2\lambda+b_3, \\ z&=c_0\lambda^3+c_1\lambda^2+c_2\lambda+c_3. \end{aligned}\right\} \cdots \qquad \cdots \quad (1)$$

The implicit analytical equation of the curve is obtained by eliminating λ between the equations (1). The resulting equation is of degree three and therefore it represents a cubic curve.

The equation of the line which joins two points on the curve whose parameters are respectively α and β can be obtained in the form—

$$\begin{vmatrix} x & a_0 & a_1 & a_2 & a_3 \\ y & b_0 & b_1 & b_2 & b_3 \\ z & c_0 & c_1 & c_2 & c_3 \\ 0 & 1 & -(\alpha+\beta) & \alpha\beta & 0 \\ 0 & 0 & 1 & -(\alpha+\beta) & \alpha\beta \end{vmatrix}=0 \qquad \cdots \quad (2)$$

If we put $\alpha=\beta$, we obtain the equation of the tangent at the point α in the form

$$\begin{vmatrix} x & a_0 & a_1 & a_2 & a_3 \\ y & b_0 & b_1 & b_2 & b_3 \\ z & c_0 & c_1 & c_2 & c_3 \\ 0 & 1 & -2\alpha & \alpha^2 & 0 \\ 0 & 0 & 1 & -2\alpha & \alpha^2 \end{vmatrix}=0 \qquad \ldots \quad (3)$$

From this it follows that the curve is of the fourth class, since this equation is of degree four in α.

220. We have shown that the equation of all autotomic cubics can be reduced to the form

$$(y^2-kx^2)z=x^3 \quad \ldots \qquad \ldots \quad (1)$$

The third vertex C is a node, a cusp, or a conjugate point according as $k >, =, < 0$.

Now, if we put $x=\lambda y$ in the above equation, we obtain the following parametric representation of the curve:—

$$\left.\begin{aligned} \rho x &= \lambda\,(1-k\lambda^2) \\ \rho y &= 1-k\lambda^2 \\ \rho z &= \lambda^3 \end{aligned}\right\} \quad \ldots \qquad \ldots \quad (1)$$

The equation of the line joining any two points on the curve whose parameters are α and β is

$$x(\alpha^2+\alpha\beta+\beta^2-k\alpha^2\beta^2)-y\alpha\beta(\alpha+\beta)$$
$$=(1-k\alpha^2)(1-k\beta^2)z \quad \ldots \quad (2)$$

and therefore the equation of the tangent at the point α is

$$x\,(3\alpha^2-k\alpha^4)-2\alpha^3y=(1-k\alpha^2)^2z.$$

or, $$\alpha^2\{x(3-k\alpha^2)-2\alpha y\}=(1-k\alpha^2)^2z. \qquad \ldots \quad (3)$$

If the tangent passes through a fixed point, this equation gives the parameters of the points of contact

of the four tangents, which can be drawn from the point. Let a_1, a_2, a_3, a_4 be the four values of the parameter in (3). Now, since the coefficient of a is absent,

we have $\sum a_1 a_2 a_3 = o$ *i.e.* $\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}=o.$

221. From equation (1) of the last article, it follows that the parameter for the double point is given by $\lambda = \pm\sqrt{1/k}$. If the curve has a node, $k > o$, and λ has two distinct real values; and consequently the curve consists of two branches, of which one corresponds to the values of the parameter between $+\sqrt{1/k}$ and $-\sqrt{1/k}$ through 0, and the other between $+\sqrt{1/k}$ and $-\sqrt{1/k}$ through ∞. As k becomes smaller and smaller, the second branch gradually disappears and finally forms a loop at the double point. If $k=0$, there is a cusp produced by the gradual contraction of the loop, when the parameter passes through ∞. If k is negative, the double point becomes a conjugate point and it no longer lies on the continuous branch whose points correspond to the real values of the parameters from $-\infty$ to $+\infty$. This acnode is an isolated point that cannot be included in the description of the curve by a real tracing point. Hence a unicursal curve consists of a single circuit, *i.e.* it is *unipartite*.

222. The fundamental property of a unicursal curve is that the coordinates of any point of it can be expressed as rational and integral functions of a variable parameter. Thus, by giving to this parameter values continually increasing from negative to positive infinity, the coordinates become all real, and all the points on the curve are obtained in a continuous series. Therefore, a unicursal curve consists of a singe circuit with the real points of it arranged in one continuous series,

i.e. it is unipartite. But all unipartite curves are not necessarily unicursal. For there are unipartite curves, the coordinates of whose points cannot be rationally expressed in terms of a single parameter. A crunodal cubic is unicursal and unipartite, all the points on the curve succeed each other in a definite order, and form a single circuit. The curve, as we have seen above, may be regarded as consisting of a loop and an infinite branch the two parts of which are separated by the loop. A cuspidal cubic and an acnodal cubic are each of them unicursal and unipartite.

223. When the curve consists of an oval and an infinite branch, every right line, which meets the oval once, must meet it a second time and no more. Therefore, that line can meet the infinite branch only once. It follows hence that no tangent to the curve can meet the oval again, and thus no points of inflexion can lie on the oval part. But from any point outside the oval, two tangents can be drawn to it. Thus the oval is a continuous series of points from none of which any real tangent, distinct from the tangent at the point, can be drawn to the curve. The four tangents drawn from any point on the infinite branch are all real,—two to the oval and two to the branch itself, while the tangents from any point on the oval are all imaginary. The tangent *at* any point of the infinite branch must meet that branch again and not the oval, since if it meets the oval, it must meet it twice.

Again, in the present case the points are not arranged in a single circuit. For, if we start with any point on the oval and proceed continuously, we return to the point whence we started, without passing through any point on the infinite branch. In this case the coordinates

cannot be rationally expressed in terms of a single parameter. The curve consists of two distinct branches and is called a *bipartite* curve.

224. We shall next determine the condition which must be satisfied by the parameters of three collinear points on a unicursal cubic. Let $lx+my+nz=o$ be the equation of a line, meeting the curve in three points whose parameters are $\lambda_1, \lambda_2, \lambda_3$ respectively. Substituting the values of x, y, z in this from the equation (1) of § 220 we obtain

$$\lambda^3(n-lk)-\lambda^2 mk+l\lambda+m=o \qquad \dots \quad (1)$$

The three roots of this equation are therefore $\lambda_1, \lambda_2, \lambda_3$.

where $$\lambda_1+\lambda_2+\lambda_3=\frac{mk}{n-lk}$$

and $$\lambda_1\lambda_2\lambda_3=\frac{-m}{n-lk}.$$

Therefore, $$\lambda_1+\lambda_2+\lambda_3+k\lambda_1\lambda_2\lambda_3=o \qquad \dots \quad (2)$$

This relation must always be satisfied by the parameters of three collinear points on the cubic.

From equation (2) we may obtain other equations also. Thus, if b be the point of contact of a tangent drawn from a point a, we have $a+2b+kab^2=o \quad \dots \qquad \dots \quad (3)$

This equation gives in general two values of b, thus giving the parameters of the two tangents which can be drawn from the point 'a' on the nodal cubic (§ 65). On the other hand, if b is given, the equation (3) determines the tangential point 'a'.

Again, the discriminant of (3) is $\Delta\equiv 1-ka^2$. Hence it follows that if the absolute value of a is less than $\sqrt{1/k}$, b is imaginary, and consequently no real tangent can be drawn to the curve. But in this case the point lies on the oval. Hence, from a point on the oval no real tangent can be

drawn, as we have otherwise shown. If $a=\pm\sqrt{1/k}$, both the points of contact coincide with the double point. It then gives $b=\mp\sqrt{1/k}$ as the parameter of the cusp. If $k=o$, one of the roots of (3) is infinite, and the loop contracts into the cusp, and from any point on a cuspidal cubic only *one* tangent can be drawn to the curve and the parameter of this point of contact is $b_1=-\frac{1}{2}a$, the other tangent corresponds to $b_2=\infty$, which is therefore an asymptote to the curve.

225. If $a=b$, *i.e.* if the tangential point coincides with the point itself, it is a point of inflexion, and the above equation (3) becomes $3a+ka^3=o$, whose roots give the parameters of the points of inflexion. These parameters are, therefore,

$$a_1=o,\ a_2=+\sqrt{-3/k},\ a_3=-\sqrt{-3/k}.$$

These three points of inflexion are real only in the case when $k<o$, *i.e.* when the curve has an acnode. When $k>o$, the curve has a node, and we see that there is only *one* real point of inflexion at the vertex B.

Again, $a_1+a_2+a_3=o$ and $a_1a_2a_3=o$.

$\therefore\ a_1+a_2+a_3+ka_1a_2a_3=o$ *i.e.*, the three points of inflexion of a unicursal cubic lie on a right line.

226. If we draw a line through the point of inflexion B, since $\lambda_1=o$ in this case, from the above equation it follows that $\lambda_2+\lambda_3=o$, or, $\lambda_2=-\lambda_3$. Now, the lines which join these two points to the third vertex are of the forms $x-\lambda y=o$ and $x+\lambda y=o$, which are harmonic conjugates with respect to $x=o$ and $y=o$. Therefore the four points B, λ_2, λ_3 and D (D being the point in which the line meets $y=o$) form a harmonic range. Consequently, $y=o$ is the harmonic polar of the point of inflexion B. In fact, the polar conic of B is $yz=o$, $z=o$ is the inflexional

tangent and $y=o$ is the harmonic polar; $x=o$ is the line joining the point of inflexion with the double point.

227. We have seen that the equation of a nodal cubic can be reduced to the form $x^3+y^3+6mxyz=o$. Consider the line $x=\lambda y$ drawn through the double point C.

Then $y^3(1+\lambda^3)+6m\lambda y^2 z=o$, which gives

$$y/z=\frac{-6m\lambda}{1+\lambda^3}, \text{ and consequently } \frac{x}{z}=\frac{-6m\lambda^2}{1+\lambda^3}.$$

Thus, $x:y:z=-6m\lambda^2:-6m\lambda:1+\lambda^3$.

The condition that any three points $\lambda_1, \lambda_2, \lambda_3$ are collinear is obtained as before in the form $\lambda_1\lambda_2\lambda_3=-1$. If $\lambda_1=\lambda_2$, we obtain $\lambda_1{}^2=-1/\lambda_3$, or, $\lambda_3=-1/\lambda_1{}^2$.

Therefore the tangential point λ_3 of λ_1 is $-1/\lambda_1{}^2$, and the points of contact of the tangents drawn from any point λ_3 are given by $\lambda_1{}^2=-1/\lambda_3$, *i.e.* $\pm\sqrt{1/\lambda_3}$.

If the line is an inflexional tangent, $\lambda_1=\lambda_2=\lambda_3$ and the condition is $1+\lambda^3=o$, which gives the parameters of the points of inflexion.

Example:—The parametric representation of the Folium of Descartes $x^3+y^3+3axyz=o$ is given by

$$x:y:z=-3a\lambda^2:-3a\lambda:1+\lambda^3.$$

228. We have shown that the equation of all cuspidal cubics can be reduced to the form $y^2z=x^3$. This follows also from the equation of § 192 when $k=o$. The coordinates of any point on the curve can be expressed as $x:y:z=\lambda:1:\lambda^3$. The condition that three points on a cuspidal cubic should be collinear becomes $\lambda_1+\lambda_2+\lambda_3=o$. Hence, the tangential of a point λ is -2λ, and the point of contact of the tangent drawn from any point λ is $-\frac{1}{2}\lambda$. At a point of inflexion $3\lambda=o$, *i.e.* $\lambda=o$.

The equation of a line joining any two points λ_1, λ_2 on the curve is $(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)x-\lambda_1\lambda_2(\lambda_1+\lambda_2)y-z=o$. For, this is satisfied by the coordinates of the two points having parameters λ_1 and λ_2. Hence the equation of the tangent at a point $\lambda=\lambda_1=\lambda_2$ is

$$3\lambda^2x-2\lambda^3y-z=o.$$

If this tangent passes through a fixed point, the above equation in λ gives the values of the parameters of the points, the tangents at which pass through the fixed point. But in this equation the coefficient of λ is absent, and therefore, if λ_1, λ_2, λ_3 be the parameters, we have

$$\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1=o, \text{ i.e. } \Sigma\frac{1}{\lambda}=o.$$

Hence the parameters of the points, the tangents at which meet in a point, are connected by the relation

$$\Sigma\frac{1}{\lambda}=o.$$

229. In general, if $f(x, y, z)=o$ be the equation of a curve of the nth degree, the points of intersection of the cubic with this curve are given by $f(\lambda, 1, \lambda^3)=o$. This last equation wants the term λ^{3n-1}, and therefore the sum of the roots $=o$. Thus the parameters of the points of intersection of a cubic with a curve of the nth degree are connected by the relation $\Sigma\lambda=o$.

Thus, if a conic intersects the cubic in six points whose parameters are a, b, c, d, e, f, then $a+b+c+d+e+f=o$. Again, if g is a point collinear with e and f on the curve, $e+f+g=o$. $\therefore g=-(e+f)=a+b+c+d$. Hence follows the theorem of § 155 that if a conic through four points on a cubic intersects this latter in two other points, the line joining them passes through a fixed point on the cubic. Similarly, all other theorems in residuation follow in the case of a cuspidal cubic.

Examples.

1. Express in terms of a single parameter the coordinates of a point on the curve $y^2z=x^2(x-z)$.

The equation is $(x^2+y^2)\ z=x^3$

$\therefore$ By § 220, the coordinates of a point are given by

$$x=\lambda\ (1+\lambda^2)$$

$$y=(1+\lambda^2)$$

$$z=\lambda^3.$$

2.* Through a given point on a cuspidal cubic a line is drawn. Find the locus of the intersection of the tangents at the other two points where the line intersects the cubic.

Let the cubic be $y^2z=x^3$, so that the coordinates of a point are $x:y:z=\lambda:1:\lambda^3$.

Let the parameter of the given point be λ_3, and those of the other two points on a line through this point be λ_1 and λ_2. Then $\lambda_1+\lambda_2+\lambda_3=o$ ... ... (1)

The tangents at λ_1 and λ_2 are

$$3\lambda_1^2x-2\lambda_1^3y-z=o \quad \ldots \quad \ldots \quad (2)$$

$$3\lambda_2^2x-2\lambda_2^3y-z=o \quad \ldots \quad \ldots \quad (3)$$

To find the locus of the intersection of (2) and (3), we have to eliminate λ_1 and λ_2 between (1), (2) and (3).

Subtracting (3) from (2) we obtain

$$3(\lambda_1+\lambda_2)\ x-2\ (\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)y=o,$$

or, $$3\lambda_3x+2\{(\lambda_1+\lambda_2)^2-\lambda_1\lambda_2\}y=o,$$

or, $$3\lambda_3x+2\lambda_3^2y-2\lambda_1\lambda_2y=o,$$

or, $$\lambda_3(3x+2\lambda_3y)-2\lambda_1\lambda_2y=o$$

i.e. $$\lambda_3(3x+2\lambda_3y)^2=4\lambda_1^2\lambda_2^2y^2/\lambda_3 \quad \ldots \quad (4)$$

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* Salmon—H. P. Curves *Ex.* 1, § 213.

Again, from (2) and (3), $2\lambda_1^2\lambda_2^2 y/\lambda_3 = -z$.

$\therefore$ From (4), the required locus is $\lambda_3(3x+2\lambda_3 y)^2+2yz=o$, which is a conic.

230. *A unicursal cubic possesses one or three real points of inflexion, according as it has a node, or a conjugate point. If it has a cusp, it has always a single real point of inflexion.*

Let the curve be defined by

$$\left.\begin{aligned} x &= a_0\lambda^3+a_1\lambda^2+a_2\lambda+a_3 \\ y &= b_0\lambda^3+b_1\lambda^2+b_2\lambda+b_3 \\ z &= c_0\lambda^3+c_1\lambda^2+c_2\lambda+c_3 \end{aligned}\right\} \quad \ldots \qquad \ldots \quad (1)$$

Let us find the condition that the three points whose parameters are $\lambda_1, \lambda_2, \lambda_3$ respectively are collinear. Substituting the coordinates of the three points in the equation of a line and eliminating the coefficients, we obtain a determinant which is the product of the matrices

$$\begin{Vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \end{Vmatrix} \times \begin{Vmatrix} \lambda_1^3 & \lambda_1^2 & \lambda_1 & 1 \\ \lambda_2^3 & \lambda_2^2 & \lambda_3 & 1 \\ \lambda_3^3 & \lambda_3^2 & \lambda_3 & 1 \end{Vmatrix} = 0$$

which after simplification becomes

$$\begin{vmatrix} 1 & -(\lambda_1+\lambda_2+\lambda_3) & \lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1 & -\lambda_1\lambda_2\lambda_3 \\ a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \end{vmatrix} = 0 \quad (2)$$

This can be written as

$$A_0+A_1(\lambda_1+\lambda_2+\lambda_3)+A_2(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)$$
$$+A_3\lambda_1\lambda_2\lambda_3=0. \quad \ldots \qquad \ldots \quad (3)$$

where A_0, A_1, A_2, A_3 represent the determinants of the matrix

$$\begin{Vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \end{Vmatrix}.$$

Now, if we put $\lambda_1 = \lambda_2 = \lambda_3$, the equation (3) becomes

$$A_0 + 3A_1\rho + 3A_2\rho^2 + A_3\rho^3 = o \qquad \ldots \quad (4)$$

whose roots are the parameters of the three points of inflexion.

The discriminant of (4) is

$$D \equiv (A_0A_3 - A_1A_2)^2 - 4(A_0A_2 - A_1^2)(A_1A_3 - A^2) \quad \ldots \quad (5).$$

The roots of the equation (4) will be one or all three real, according as D is positive or negative.

Again, a double point on the cubic corresponds to two different values of the parameter (say) α and β. If γ be any point on the curve, then $A_0 + A_1(\alpha+\beta+\gamma) + A_2(\alpha\beta+\beta\gamma+\gamma\alpha) + A_3\alpha\beta\gamma = o$ must be satisfied. This may be written as

$$\{A_0 + A_1(\alpha+\beta) + A_2\alpha\beta\} + \gamma\{A_1 + A_2(\alpha+\beta) + A_3\alpha\beta\} = o.$$

This relation must be satisfied for all values of γ, since the line always passes through the double point, and α and β are fixed.

$$\therefore \qquad \left.\begin{aligned} A_0 + A_1(\alpha+\beta) + A_2\alpha\beta = o \\ A_1 + A_2(\alpha+\beta) + A_3\alpha\beta = o \end{aligned}\right\} \qquad \ldots \quad (6)$$

Again α and β, the parameters for the double point are the roots of an equation $\rho^2 - (\alpha+\beta)\rho + \alpha\beta = o$ ... (7)

Combining (6) and (7) we find that α and β are the roots of

$$\begin{vmatrix} 1 & -\rho & \rho^2 \\ A_0 & A_1 & A_2 \\ A_1 & A_2 & A_3 \end{vmatrix} = 0. \qquad \ldots \quad (8)$$

The discriminant of (8) is given by

$$D \equiv (A_0A_3 - A_1A_2)^2 - 4(A_0A_2 - A_1^2)(A_1A_3 - A_2^2)$$

Therefore the roots of (8) are real or imaginary, and consequently the double point is a node or a conjugate point, according as D is positive or negative. Therefore, from what has been said above, it follows that the curve has a node or a conjugate point according as D is positive or negative, which again is the condition for one or three real points of inflexion. Hence the truth of the theorem follows.

If $D=o$, the cubic has a cusp, the curve has only one point of inflexion and that must be a real one.

CHAPTER XIV.

SPECIAL CUBICS.

231. **Definition :**—A circular cubic is a cubic curve which passes through the two circular points at infinity.

Properties of circular cubics are best studied in connection with those of bicircular quartics; for they may be regarded as degenerate forms of the latter. In the present chapter, we shall study their properties as inverse of conics with respect to a point on the curve.

232. **The Equation of a Circular Curve:**

Let $C=o$ be the equation of a circle and $S=o$ the equation of a conic. Then if $u_1=o$ be the equation of a right line, the equation of a circular cubic is $u_1 C + k S = o$ ... (1) where $k=o$ is the line at infinity. For, this is a cubic curve and evidently passes through the points where the line at infinity intersects the circle, *i.e.* the two circular points at infinity. Again, since the line $u_1=o$ meets the curve at one point at infinity and two points at a finite distance *i.e.* on the conic S, the line u_1 is parallel to an asymptote of the curve.

The general Cartesian equation of a circular cubic can therefore be written as

$$(lx+my+n)\,(x^2+y^2+2gx+2fy+c)+S=o, \qquad \ldots \quad (2)$$

where $S=o$ is the general equation of a conic.

or, in the form $(x^2+y^2)\,(lx+my)+V=o$ ... (3)

where V is a conic.

The Trilinear equation can be written as

$$(lx+my+nz)(ayz+bzx+cxy)+(ax+by+cz)S=o \quad \ldots \quad (4)$$

where S is any conic.

233. *The inverse of a conic with respect to a point on the curve is a circular cubic, whose real asymptote is parallel to the tangent to the conic at the centre of inversion.*

The equation of a conic having the origin on the curve is of the form $ax^2+by^2+2hxy+2fy+2gx=0$ ... (1)

The inverse of this with respect to the origin is—

$$\frac{k^4}{r^4}(ax^2+2hxy+by^2)+\frac{k^2}{r^2}(2fy+2gx)=0;$$

or, $r^2(2fy+2gx)+k^2(ax^2+2hxy+by^2)=0.$ ... (2)

Comparing this with equation (1) of the preceding article, it is seen that the inverse is a circular cubic and $2fy+2gx=0$, which is the tangent to the conic at the origin, is parallel to the real asymptote of the cubic, which proves the theorem.

It is evident from (2) that the origin is a double point on the cubic, which is a node, a cusp, or a conjugate point, according as the conic is an hyperbola, a parabola, or an ellipse.

Conversely, the inverse of a circular cubic with regard to the double point is a conic through the double point.

The equation of a conic with regard to a vertex is

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}-\frac{2x}{\alpha}=0.$$

The inverse of this with regard to the origin is

$$x(x^2+y^2)=ax^2+by^2 \qquad \dots \qquad \dots \quad (3)$$

where $a=\frac{1}{2}k^2/\alpha, \qquad b=\frac{1}{2}k^2\alpha/\beta^2.$

234. *To find the equation of a circular cubic which has two imaginary points of inflexion at the two circular points.*

The equation of a cubic having three points of inflexion at infinity may be written as $uvw+\mathrm{I}^3=0$ ... ... (1) where u, v, w are the inflexional tangents. If the circular

points are points of inflexion on the curve, we may take the circular lines through the origin as tangents at those two points, so that $vw \equiv x \pm iy = x^2 + y^2$.

Therefore, the equation (1) may be written as—

$$u(x^2+y^2) + I^3 = 0, \text{ which is a circular cubic;}$$

i.e. $(x^2+y^2)(lx+my+n)+c^3=0$, where l, m, n, c are constants.

The line $u \equiv lx+my+n=0$ is the tangent to the curve at the real point of inflexion which is also at infinity.

235. In §233 we have seen that the inverse of a conic with regard to a vertex is $x(x^2+y^2)=ax^2+by^2$ ... (1)

The following cases are to be considered:—

(1) When a and b have the same sign, the curve is the inverse of an ellipse, and when they are of opposite signs, it is the inverse of an hyperbola.

(2) When $a=0$, the cubic is the inverse of a parabola and is called a *Cissoid.*

(3) When $a+b=0$, the curve is the inverse of a rectangular hyperbola and is called the *Logocyclic cubic.*

The curve (1) has a double point at the origin, the tangents at which are $ax^2+by^2=0$. Therefore, the origin is a node, a cusp, or a conjugate point, according as $ax^2+by^2=0$ represents two real and distinct, coincident, or imaginary lines; *i.e.* according as the conic is an hyperbola, a parabola, or an ellipse. The curve again cuts the axis of x at the point $x=a$, which is called the vertex. The only real asymptote to the curve is $x=b$, and the curve has a real point of inflexion at infinity.

236. *The Logocyclic curve is the inverse of a rectangular hyperbola with regard to a vertex.*

The Cartesian equation of the curve is—

$$x(x^2+y^2)=a(x^2-y^2) \quad \ldots \quad \ldots \quad \ldots (1)$$

The nodal tangents are $x^2 - y^2 = 0$, which are two right lines at right angles to each other. The only real asymptote is $x + a = 0$, which, therefore, makes an angle of $45°$ with each nodal tangent. (Fig. 27.)

The polar equation of the curve is $r\cos\theta = a\cos 2\theta$.

Let O be the origin and A the vertex, and let the axis of x cut the asymptote at B. Transfering the origin to the vertex A, the polar equation of the curve becomes

$$r^2 + 2ar\sec\theta + a^2 = 0 \quad \ldots \quad \ldots \quad (2)$$

or $$r = -a(\sec\theta \pm \tan\theta) \quad \ldots \quad \ldots \quad (3)$$

or, changing the constant

$$r = a(\sec\theta \pm \tan\theta).$$

237. We may use these properties of the logocyclic cubic in showing that every nodal cubic can be projected into a logocyclic curve.

Consider a nodal cubic having a node at O with the nodal tangents OA and OB (Fig. 28). Let OP and OQ be two lines drawn through O, which are harmonic conjugates of OA and OB. Then, OP and OQ meet the curve each in one other point. Let the line PQ intersect the curve in a third point R, and the tangent at R meet the nodal tangents in A and B respectively. Now project the figure, so that the line PQ goes off to infinity, and at the same time, the angles OAB and OBA are each projected into an angle of $45°$. Then the projected figure will be a logocyclic curve. For, a node is projected into a node and the nodal tangents are projected into nodal tangents. The line AB is projected into the asymptote, which makes with each nodal tangent an angle of $45°$. Therefore, the angle between the nodal tangents is a right angle, and since OP and OQ are harmonic conjugates of OA and OB, and also harmonic properties are unaltered by projection, OP

and OQ are projected into the two circular lines*. ? Thus, the projected curve is a circular cubic, the points P and Q being projected into the two circular points at infinity.

Note: Properties of logocyclic cubics have been discussed by Dr. Booth in connection with the geometrical treatment of logarithms, a detailed account of which will be found in his book—Treatise on some Geometrical methods.

238. We may construct the logocyclic cubic geometrically by the following method :—

Let O be a fixed point and AB be a fixed right line. Draw OA perpendicular on AB. Let any right line OP through O intersect AB at B, and on OP take two points P, P' such that $PB=P'B=AB$. Then the locus of P and P' is the logocyclic cubic, of which O is the vertex, A the node (Fig. 29).

For, let $\angle AOP=\theta$. Then $OP=OB-PB=OB-AB$

$$=OA(sec\theta+\tan\theta)$$

$\therefore$ The polar equation of the locus is $r=a(sec\theta\pm\tan\theta)$ which is the equation of the logocyclic cubic, whose vertex is the origin and A is the node. (§ 236).

239. The Trisectrix of Maclaurin:

The equation of the Trisectrix of Maclaurin is

$$x(x^2+y^2)=a(y^2-3x^2) \qquad \ldots \qquad \ldots \qquad (1)$$

Therefore, the curve is a circular cubic, the origin being a node. The nodal tangents are $y\pm\sqrt{3}x=0$ and therefore each makes an angle of 60° with the real asymptote. The circular points are points of inflexion on the curve.

* C. V. Durell—Geometry for advanced students Vol. II. p, .31.

For, any line through I is of the form $y = \pm ix + h$ ... (2)
This line intersects (1), where

$$x\{x^2 + (\pm ix + h)^2\} = a\{(\pm ix + h)^2 - 3x^2\}$$

$$\text{or, } x\{\pm 2ih\cancel{x} + h^2\} = a\{(\pm ix + h)^2 - 3x^2\}$$

$$\textit{i.e. } x^2(\pm 2ih + 4a) + x(h^2 \mp 2iah) - ah^2 = 0 \quad \ldots \quad \ldots (3)$$

Hence, from (3) we see that one point is at infinity. Two points will be at infinity, if $\pm 2ih + 4a = 0$, *i.e.* if $h = \pm 2ia$. But then the coefficient of x also vanishes, and therefore three points of intersection coincide at infinity, if $h = \pm 2ia$. Hence the lines $y = \pm ix \pm 2ia$ are inflexional tangents to the curve, the points I and J being two imaginary points of inflexion.

Now, the line IJ intersects the curve in one other point R, which must therefore be a real point of inflexion. The tangent at this point is $x = a$. For, the equation (1) can be written as—

$$(x-a)\ (x^2 + y^2) + a(x^2 + y^2) - a(y^2 - 3x^2) = 0$$

$$\text{or, } (x-a)\ (x^2 + y^2) + 4ax^2 = 0.$$

∴ Where $x = a$ intersects the curve, we have $4a \cdot a^2 = 0$ *i.e.* $a^3 = 0$, which shows that the point $x = a$ is a point of inflexion.

240. The Trisectrix of Maclaurin may be defined as a nodal circular cubic which has two points of inflexion at the circular points.

The equation of a circular cubic having the node at the origin is $(lx + my)\ (x^2 + y^2) + ax^2 + 2hxy + by^2 = 0$ ... (1)

If the real asymptote is taken parallel to the axis of y, the equation becomes $x(x^2 + y^2) + ax^2 + 2hxy + by^2 = 0$...(2).

Then we have to determine the constants a, h, b from the condition that the circular points will be points of inflexion on the curve.

The equation of a line through I or J is of the form $y=\pm ix+k$. The abscissae of the points where this intersects (2) are given by—

$$x\{x^2+(\pm ix+k)^2\}+ax^2+2hx(\pm ix+k)+b(\pm ix+k)^2=0\,;$$

or, $$x\{\pm 2ikx+k^2\}+x^2(a-b\pm 2ih)+2x(hk\pm ibk)+bk^2=0;$$

i.e. $$x^2(a-b\pm 2ik\pm 2ih)+x(2hk\pm 2ibk+k^2)+bk^2=0 \quad \ldots(3)$$

If the line is an inflexional tangent, the equation (3) should have three roots infinite. One root is evidently infinite. The remaining two will be infinite, if

$$\left.\begin{aligned} a-b\pm 2i(h+k)&=0,\\ \text{and}\quad k(2h+k)\pm 2ibk&=0 \end{aligned}\right\}$$

$$\therefore\ 2h+k\pm 2ib=0,\quad i.e.\ k=-2(h\pm ib)$$

$$\therefore\ a-b\pm 2i(h-2h\mp 2ib)=0,\ \text{which gives}$$

$$a+3b=\pm 2ih. \qquad \therefore\ h=0,\ a+3b=0.$$

Therefore the equation (1) becomes $x(x^2+y^2)=b(3x^2-y^2)$ Putting $-a$ for b, we obtain $x(x^2+y^2)=a(y^2-3x^2)$.

Every nodal cubic can therefore be projected into a Trisectrix of Maclaurin, by projecting any two points of inflexion into the two circular points at infinity.

241. The Trisectrix of Maclaurin may be geometrically constructed by the following method :—

Let O be the centre of a circle and AOB a diameter. Through the middle point H of OA draw HC perpendicular to OA. Through A draw the line ACD cutting HC at C, and the circle at D. On DA produced, take a point P, such that AP=CD. Then the locus of P is a Trisectrix of Maclaurin. (Fig. 30).

For, let A be the origin, and the coordinates of P de (x, y). Let $\angle$ BAD$=\theta$ and AO$=2a$.

$$\therefore\ -x = \text{AM} = \text{AP}\cos\theta = \text{CD}\cos\theta = (\text{AD} - \text{AC})\cos\theta$$
$$= (2\text{AB}\cos\theta - \text{AH}\sec\theta)\cos\theta.$$
$$= (4a\cos\theta - a\sec\theta)\cos\theta.$$
$$= 4a\cos^2\theta - a$$
$$= a(4\cos^2\theta - 1) \quad \ldots \quad \ldots \quad (1).$$

and $-y = \text{PM} = \text{AP}\sin\theta = \text{CD}\sin\theta$

$$= (4a\cos\theta - a\sec\theta)\sin\theta$$
$$= a(4\cos^2\theta - 1)\tan\theta \quad \ldots \quad \ldots \quad (2)$$

Eliminating θ between (1) and (2) we obtain—
$x(x^2 + y^2) = a(y^2 - 3x^2)$, which is the Trisectrix of Maclaurin.

242. The Folium of Descartes:

The equation of the curve is $x^3 + y^3 = 3axy \quad \ldots \quad (1).$

There is symmetry about the line $y = x$, and the tangents at the origin are the axes of coordinates.

The equation of the curve can be written as

$$x^3 + y^3 + 3xy\,(x + y) = 3xy\,(x + y + a)$$

$$\text{or, } (x + y)^3 = 3xy\,(x + y + a) \quad \ldots \quad \ldots \quad (2).$$

The form (2) shows that the curve has three points of inflexion lying on the line $x + y = 0$ and the inflexional tangents are $x = 0$, $y = 0$, and $x + y + a = 0$.

But $x = 0$ and $y = 0$ are the nodal tangents at the origin and cannot therefore be regarded as inflexional tangents. The real point of inflexion is given by $x + y = 0$, $x + y + a = 0$, which lines meet at infinity. Therefore, $x + y + a = 0$ is the real asymptote to the curve; in fact, it is an inflexional tangent, the point of contact being at infinity. Again, the equation (1) can be written as—

$$x^3 + \omega^3 y^3 + 3\omega xy(x + \omega y) = 3xy(\omega x + \omega^2 y + a).$$

Or

$(x + \omega y)^3 = 3xy(\omega x + \omega^2 y + a)$, where ω is an imaginary cube root of unity.

This shows, as before, that $\omega x+\omega^2 y+a=0$ is an inflexional tangent, the point of contact being at infinity. Since ω or ω^2 may be regarded as a cube root of unity, we obtain $\omega^2 x+\omega y+a=0$ as another inflexional tangent, whose point of contact also lies at infinity. Hence, all the three points of inflexion lie at infinity.

The real asymptote is the line $x+y+a=0$, which evidently makes an angle of 45° with each of the axes, *i.e.* the nodal tangents. The form of the curve is almost similar to that of the logocyclic cubic and is shown in Fig. 31. Hence the Folium of Descartes is a nodal cubic, whose three points of inflexion lie at infinity and the tangent at one of these points makes an angle of 45° with the nodal tangents.

Thus, every nodal cubic can be projected into the Folium of Descartes. For, let ABC be the line on which lie the three points of inflexion, and let the tangent at the real point of inflexion cut the nodal tangents at P and Q, O being the node. Project the figure such that ABC goes off to infinity and the angles OPQ and OQP are projected each into an angle of 45°. Then the nodal tangents of the projected figure are mutually orthogonal and the points of inflexion are at infinity.

243. If the axes be turned through an angle of 45°, the equation of the curve becomes—

$$x(x^2+3y^2)=a(x^2-y^2) \qquad \ldots \qquad \ldots \qquad (3)$$

The form (3) suggests the following method of generating the curve geometrically.

Let O be the centre of a circle, OA and OB being two perpendicular diameters. (Fig 32.) Draw ABC any line intersecting OB at B and the circle at C. On AB take a point Q, such that AQ=BC. Let P be the harmonic

conjugate of Q with regard to A and B. Then the locus of P is a Folium of Descartes.

Let $OA=a, \angle PAM=\theta$, and P be the point (x, y).

Then, $x=AM=AP\cos\theta$, and $y=AP\sin\theta$.

Also, $$\frac{1}{AP}+\frac{1}{AB}=\frac{2}{AQ} \qquad \ldots \qquad \ldots \qquad (4)$$

But $$AB=a\sec\theta.$$

$\therefore\ AQ=BC=AC-AB=2a\cos\theta-a\sec\theta.$

$$\therefore \frac{1}{AP}=\frac{2}{2a\cos\theta-a\sec\theta}-\frac{1}{a\sec\theta}$$

$$=\frac{2a\sec\theta-2a\cos\theta+a\sec\theta}{a\sec\theta\,(2a\cos\theta-a\sec\theta)}$$

$$\therefore\ AP=\frac{a\sec\theta\,(2a\cos\theta-a\sec\theta)}{a\sec\theta\,(3-2\cos^2\theta)}$$

$$=\frac{a\cos 2\theta}{\cos\theta\,(3-2\cos^2\theta)}$$

$$\therefore\ x=AP\cos\theta=\frac{a\cos 2\theta}{3-2\cos^2\theta} \qquad \ldots \qquad \ldots \qquad (5)$$

and $$y=AP\sin\theta=\frac{a\cos 2\theta}{3-2\cos^2\theta}\cdot\tan\theta \qquad \ldots \qquad \ldots \qquad (6)$$

Eliminating θ between (5) and (6) we obtain——

$a(x^2-y^2)=x(x^2+3y^2)$, which is the Folium of Descartes.

The locus of Q is the logocyclic cubic; for, if (x, y) be the coordinates of Q,

$$\frac{x^2-y^2}{x^2+y^2}=\cos 2\theta=\frac{AQ\cos\theta}{a}=\frac{x}{a}$$

$\therefore\ x(x^2+y^2)=a(x^2-y^2)$, which is the logocyclic cubic.

244. The Cissoid :

The Cissoid is the inverse of a parabola with respect to its vertex. If in the equation (3) of § 233, we put $a=0$, the equation of the Cissoid becomes $x(x^2+y^2)=by^2$ ... (I).

If the equation of a parabola is $y^2=4ax$, its inverse with respect to the vertex is $x(x^2+y^2)=\dfrac{k^2}{4a}y^2$.

$$\text{or, } x(x^2+y^2)=by^2, \text{ where } b=\frac{k^2}{4a}.$$

This curve is also the first positive pedal of the parabola $y^2+4bx=0$ with respect to its vertex. For, if $x\cos\omega+y\sin\omega=p$ is a tangent to the parabola,

$$p\cos\omega=b\sin^2\omega.$$

∴ The polar equation of the pedal is $r\cos\theta=b\sin^2\theta$, from which we obtain the Cartesian equation $x(x^2+y^2)=by^2$, which is the Cissoid.

The origin is a cusp on the curve, the cuspidal tangent being $y=0$. The real asymptote of the curve is $x=b$.

The Hessian is $xy^2=0$, which shows that $x=0$ and $y=0$ are inflexional tangents. But the origin being a cusp, it cannot be regarded as a point of inflexion. Hence, the only point of inflexion lies at infinity on the line $x=0$. $x=b$ is consequently an inflexional tangent, and it cuts the cuspidal tangent at a right angle. Also the line joining the cusp to the point of inflexion and the cuspidal tangent are harmonic conjugates of the two circular lines through the cusp.

The equation of the curve can be written in the homogeneous form as

$$4(x-bz)\{y^2+(x+\tfrac{1}{2}bz)^2\}+b^2(3x+bz)z^2=0,$$

which shows that the real asymptote is $x=b$, and the two imaginary asymptotes are given by $y^2+(x+\frac{1}{2}b)^2=0$, *i.e.* a system of concentric circles, whose centre is the point $y=0$, $x+\frac{1}{2}b=0$, has double contact with the curve at the two circular points.

Hence the Cissoid is a cuspidal circular cubic, whose point of inflexion is at infinity, and the inflexional tangent is the real asymptote, which makes a right angle with the cuspidal tangent.

Therefore, every cuspidal cubic can be projected into a Cissoid. For, let O be the cusp, OB the cuspidal tangent, and A the point of inflexion with AB the inflexional tangent. Let OP and OQ, two lines through the cusp O, be harmonic conjugates of OA and OB, meeting the curve at P and Q. Then A,P,Q are collinear, for, the two harmonic pencils O(APBQ) and B(APOQ) have the self-corresponding ray OB*. Now, project P and Q into the two circular points at infinity. Then A will be projected into the point of inflexion at infinity, and the inflexional tangent will be at right angles to the cuspidal tangent.

245. The Cissoid may be geometrically constructed as follows :—

Let AOB be a circle, AB being the diameter. Let AC,BD be the tangents at A and B respectively. Let any line OA cut the circle at O, and the tangent BD at D. Take a point P upon AO, such that $AP=OD$. The locus of P will be a Cissoid. (Fig 33).

For, take AB and AC as axes of x and y respectively ; and let $AB=b$ and (x, y) be the coordinates of P, and $\angle PAM=\theta$.

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* Reye—Geometry of Position §. 86.

Then, $AP=OD=AD-AO=b\sec\theta-b\cos\theta$.

$\therefore x=AP\cos\theta=b-b\cos^2\theta$; or, $b-x=b\cos^2\theta$.

And, $y=AP\sin\theta=b\tan\theta-b\sin\theta\cos\theta$

$$\therefore y^2(b-x)=b^3(\tan\theta-\sin\theta\cos\theta)^2\cos^2\theta$$
$$=b^3(\sin\theta-\sin\theta\cos^2\theta)^2=b^3\sin^2\theta(1-\cos^2\theta)^2$$
$$=b^3(1-\cos^2\theta)^3=(b-b\cos^2\theta)^3=x^3.$$

$\therefore\ x(x^2+y^2)=by^2$, which is the Cissoid.

Newton's Method :

Newton has given the following elegant construction for the description of this curve by continuous motion:

A right angle has the arm BC of constant length b, the point C moves along a fixed line CD, while the other arm AB passes through a fixed point A at a distance equal to BC from CD. Then the locus of the middle point P of BC is a Cissoid.* (Fig 34)

Take the mid-point O of AD as origin and $\angle BAD=\theta$.

Then, $x=OM=OD-MD$. $\therefore\ MD=OD-x=\frac{1}{2}b-x$.

But $MD=PC\sin\theta=\frac{1}{2}b\sin\theta$. $\therefore\ \frac{1}{2}b-x=\frac{1}{2}b\sin\theta$...(1)

Again, $\frac{1}{2}b=BP=HK=HM+MK$

$$=(x+\tfrac{1}{2}b)\sin\theta+y\cos\theta \qquad \ldots \qquad \ldots(2)$$

Eliminating θ between (1) and (2), we obtain

$$x(x^2+y^2)=by^2, \text{ which is the Cissoid.}$$

Note: The Cissoid was invented by the Greek Geometer Diocles in the sixth century, and is better known as the Cissoid of Diocles. He made use of this curve in the geometrical construction of two mean proportionals between two given lines, *i.e.*, for solving the famous problem of the duplication of the cube.

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* Lardner's Algebraic Geometry, pp. 196, 472.

246. The Cubical Parabola :

The equation of the cubical parabola is $x^3=a^2y$. The equation shows that the origin is a point of inflexion on the curve, the tangent being $y=0$. The above equation is satisfied by $x^3=0$ and $a^2=0$. But $a^2=0$ denotes the line at infinity counted twice. Hence, the line at infinity passes through three consecutive points where $x=0$ meets it. Another line $a=0$ meets the curve in three consecutive points near the same point. Hence the point at infinity on the curve lying on $x=0$ is a cusp, and the line at infinity is the cuspidal tangent and therefore is the real asymptote.

Hence, every cuspidal cubic can be projected into a cubical parabola and this is done by projecting the cuspidal tangent to infinity.

The curve is its own reciprocal with respect to the point of inflexion.

247. The Semi-cubical Parabola :

The equation of the semi-cubical parabola is $ay^2=x^3$.

The origin is evidently a cusp, with $y=0$ as the cuspidal tangent. The directions of the asymptotes are given by $x=0$ taken thrice. This, therefore, gives the points of contact A, B, C at infinity and these are three consecutive points. Therefore the line at infinity is an inflexional tangent, the point at infinity on the curve in the direction $x=0$ being a point of inflexion. Thus the line $x=0$ joining the cusp to the point of inflexion at infinity is perpendicular to the cuspidal tangent.

Thus, a semi-cubical parabola is a cuspidal cubic, having the line at infinity for the inflexional tangent and the cuspidal tangent perpendicular to the line joining the cusp to the point of inflexion.

Hence every cuspidal cubic can be projected into a semi-cubical parabola, by projecting the inflexional tangent to infinity and at the same time projecting the angle between the cuspidal tangent and the line joining the cusp to the point of inflexion into a right angle.

248. The semi-cubical parabola is the evolute of a parabola.

For, the equation of a normal to the parabola $y^2=4ax$ is—

$$tx+y-at^3-2at=0 \;\ldots\; (1)$$

Differentiating this with respect to t, we obtain

$$3at^2=x-2a \;\ldots\ldots\ldots\ldots\; (2)$$

From (1), $2t=-\dfrac{3y}{x-2a}$ and $\therefore\ 27\,ay^2=4\,(x-2a)^3$.

Now put X for $\dfrac{x-2a}{3}$ and Y for $\dfrac{y}{2}$, and the equation becomes $aY^2=X^3$, which is the semi-cubical parabola.

This curve is the reciprocal polar of a Cissoid with respect to its cusp.

249. Foci of Circular Cubics.

Foci of circular cubics will be studied in connection with those of bicircular quartics. For the present, we notice that a nodal cubic is of class four, and circular cubics pass through the two circular points at infinity. Therefore, only two tangents can be drawn from each circular point to the curve and consequently the number of real single foci is two. The tangents at the circular points intersect at a point which is a double focus (or a quadruple focus, if we count also the imaginary foci). The tangent at I intersects the two tangents from J in two points, which are also real single (or double imaginary)

foci. Similarly the points where the tangent at J intersects the two tangents from I are real single foci. Thus the real single foci are four in number and real double focus is one.

In cuspidal circular cubics, the class is three and the cusp replacing the node, the number of real single foci is the same as in a nodal circular cubic. We shall study this more fully in a subsequent chapter.

CHAPTER XV.

INVARIANTS AND COVARIANTS OF CUBIC CURVES.

250. We shall write the general equation in three variables of a cubic in the following form :—

$$ax^3+by^3+cz^3+3fy^2z+3gz^2x+3hx^2y+3iyz^2+3jzx^2+3kxy^2+6mxyz=0 \quad \dots \quad \dots \quad (1)$$

We have shown (§ 181) that the equation of all cubics can be reduced to the semi-canonised form

$$ax^3+by^3+cz^3+6m'xyz=0 \quad \dots \quad \dots \quad (2)$$

and also to the fully canonised form

$$x^3+y^3+z^3+6mxyz=0 \quad \dots \quad \dots \quad (3)$$

From the form (3) it follows that the cubic cannot have more than two independent invariants : for if it had one more, it would have two absolute invariants.* These absolute invariants would be two functions of the co-efficients and consequently two functions of m only. By eliminating m between them, we could obtain a relation involving only the coefficients in the general equation, which is impossible, since the co-efficients in (1) are all independent.

These two independent invariants of a cubic are of degrees four and six respectively, and are denoted by S and T. All other invariants can be expressed in terms of these two only.

Note : For a general discussion of the theory and of the different methods of finding the invariants of a cubic, students are referred to Elliot's Algebra of Quantics Chap. XVI and to Salmon's Higher Plane Curves Chap. III, §§217 etc.

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* E. B. Elliot—Algebra of Quantics. §29.

251. It is shown in works on Algebra* that if in the contravariant $\phi(\lambda, \mu, \nu)$ of a quantic U, symbols of differentiation with respect to the variables are substituted for λ, μ, ν respectively, and the new function $\phi\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)$ operates upon the given quantic U, the result will be an invariant of U (Sylvester).

Now the Cayleyan is a contravariant of the cubic and expresses the condition that the line $\lambda x + \mu y + \nu z = 0$ shall be cut in involution (§ 205) by the system of polar conics

$$\frac{dU}{dx}=0, \quad \frac{dU}{dy}=0, \quad \frac{dU}{dz}=0.$$

The equation of the Cayleyan can be written as

$$C \equiv A\lambda^3 + B\mu^3 + C\nu^3 + 3F\mu^2\nu + 3G\nu^2\lambda + 3H\lambda^2\mu + 3I\nu^2\mu + 3J\nu\lambda^2 + 3K\lambda\mu^2 + 6M\lambda\mu\nu = 0 \quad \ldots \quad (4)$$

where $A, B, C, F, G, H, I, J, K, M$ are calculated by the method of § 388 (a) Salmon's Conic Sections. The values of these co-efficients are given in Salmon's Higher Plane Curves § 219.

By putting $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ for λ, μ, ν respectively in (4), and operating on the general cubic (1), we obtain an Invariant S of the fourth degree in the co-efficients.

$$\text{Thus, } S = \left[A\left(\frac{d}{dx}\right)^3 + B\left(\frac{d}{dy}\right)^3 + C\left(\frac{d}{dz}\right)^3 \right.$$

$$+ 3F\left(\frac{d}{dy}\right)^2\left(\frac{d}{dz}\right) + 3G\left(\frac{d}{dz}\right)^2\left(\frac{d}{dx}\right) + 3H\left(\frac{d}{dx}\right)^2\left(\frac{d}{dy}\right)$$

$$+ 3I\left(\frac{d}{dz}\right)^2\left(\frac{d}{dy}\right) + 3J\left(\frac{d}{dx}\right)^2\left(\frac{d}{dz}\right) + 3K\left(\frac{d}{dy}\right)^2\left(\frac{d}{dx}\right)$$

$$\left. + 6M\left(\frac{d}{dx}\right)\left(\frac{d}{dy}\right)\left(\frac{d}{dx}\right)\right] U$$

* Salmon—Higher Algebra, § 139.

$$=(ag-j^2)f^2-(ac-gj)\,kf+(cj-g^2)\,k^2+ch^2f$$

$$-hm(3fg+ck)+(hi+2m^2)\,(jf+gk)$$

$$-mi\,(af+3jk)+i^2ak$$

$$-b\{h(cj-g^2)-m(ac-gj)+i(ag-j^2)\}$$

$$-(hi-m^2)^2.$$

252. The calculation of the coefficients, and therefore of the invariant, becomes simpler, if we use the canonical form (3) of the equation.

The Cayleyan of (3) is, as we have obtained in §205,

$$C=m(\lambda^3+\mu^3+\nu^3)+(1-4m^3)\lambda\mu\nu=0.$$

Therefore, the invariant S of the canonical form is given by $S=\left[m\left\{\left(\frac{d}{dx}\right)^3+\left(\frac{d}{dy}\right)^3+\left(\frac{d}{dz}\right)^3\right\}\right.$

$$\left.+(1-4m^3)\left(\frac{d}{dx}\right)\left(\frac{d}{dy}\right)\left(\frac{d}{dz}\right)\right]\,(x^3+y^3+z^3+6mxyz)$$

$$=18m+(1-4m^3)6m$$

$$=24(m-m^4).$$

Thus, the value of S for the canonical form, removing a numerical factor is $m(1-m^3)$; and this is equal to the invariant S of the untransformed cubic (1), multiplied by the fourth power of the modulus of transformation.

From this it follows that if $S=0$, we have $m=0$, and the equation of the cubic becomes $x^3+y^3+z^3=0$.

Hence, the condition that the equation of a cubic may be expressed as the sum of three cubes, and consequently

its Hessian may reduce to three right lines is that the invariant S vanishes.

253. The second invariant T of a cubic is of degree six, and can be found by a method similar to that used in finding S. This invariant can also be obtained by using the Hessian, which is a covariant of the cubic.

It is proved in works on Algebra that an invariant of a quantic and a covariant is an invariant of the quantic itself, and also that if $U \equiv ax^3+by^3+\ldots\ldots$and $V \equiv a'x^3+b'y^3+c'z^3+\ldots\ldots$are any two quantics of the same order, and I is any invariant of U, then

$$\left(a' \frac{d}{da} + b' \frac{d}{db} + \ldots\ldots \right) I$$

is an invariant of U and V.*

Now, the Hessian of a cubic is also a cubic and a covariant. Hence, any invariant of the cubic and its Hessian is an invariant of the cubic itself. Therefore, we may calculate T by the application of the above principle to the cubic and its Hessian, both of degree three, and the invariant S. Thus the degree of the invariant T will be six.

Let the Hessian of the general cubic (1) be written in the form $a'x^3+b'y^3+c'z^3+3f'y^2z+3g'z^2x+3h'x^2y+3iyz^2$ $3j'zx^2+3kxy^2+6m'xyz=0$. ... ... (5)

We may calculate the co-efficients in (5) from the equation of the Hessian $H \equiv f_{11} \cdot f_{22} \cdot f_{33} + 2f_{12} \cdot f_{23} \cdot f_{31} - f_{11} \cdot f_{23}{}^2 - f_{22} \cdot f_{31}{}^2 - f_{33} \cdot f_{12}{}^2 = 0$. These have been given by Salmon in his Higher Plane Curves § 218.

Then, $\left(a' \dfrac{d}{da} + b' \dfrac{d}{db} + c' \dfrac{d}{dc} + \ldots \right) S$ is an invariant of the

* Elliot—loc. cit. § 19.

cubic and has been given by Salmon § 221. We do not give here the lengthy form of T on account of the tediousness of calculation.

254. The calculation of the invariant T becomes simpler for the canonical form of the cubic.

The semi-canonised cubic is $a'x^3+b'y^3+c'z^3+6m'xyz=0$. Its Hessian is $-m'^2(a'x^3+b'y^3+c'z^3)+(a'b'c'+2m'^3)xyz=0$, and the invariant $S=m'(a'b'c'-m'^3)$.

Hence, operating with

$$-m'^2\left(a'\frac{d}{da'}+b'\frac{d}{db'}+c'\frac{d}{dc'}\right)+\tfrac{1}{6}(a'b'c'+2m'^3)\frac{d}{dm'}$$

on S and multiplying the result by 6, we obtain

$$T=(a'b'c')^2-20m'^3(a'b'c')-8m'^6.$$

Therefore, by putting $a'=b'=c'=1$ and $m'=m$, we obtain the value of T for the canonical form (3) :—

Thus, $$T=1-20m^3-8m^6.$$

This invariant is not equal to the invariant T of the original cubic but to the invariant T multiplied by the sixth power of the modulus of transformation.

255. Thus we have calculated the two independent invariants S and T of a cubic, and all other invariants can be expressed as rational and integral functions of these two.*

For instance, the discriminant of the ternary cubic is $\Delta\equiv T^2+64S^3$. This can easily be calculated as follows :—

Suppose the cubic has a double point at the origin or at the third vertex C. Then the co-efficients of z^3 and z^2 in equation (1) must be zero. Therefore, $c=g=i=0$, and

* Elliot—loc. cit. § 295.

the invariant S reduces to $2m^2fj-m^4-f^2j^2$,

$$i.e. \text{ to } -(fj-m^2)^2.$$

Also, T becomes $8(f^3j^3-3m^2f^2j^2+3m^4fj-m^6)$

$$i.e.\ T=8(fj-m^2)^3$$

$$\therefore\ T^2=64\ (fj-m^2)^6=-64S^3$$

$$\therefore\ T^2+64S^3\equiv\Delta=0.$$

Thus, we see that when the cubic has a double point, $T^2+64S^3=0$, and this is the discriminant of the cubic.

If the curve have a cusp with $y=0$ as the tangent, we must have further $m=0$ and $j=0$. $\therefore$ $S=0$, $T=0$. Hence the condition, both necessary and sufficient, that the cubic has a cusp, is that both S and T vanish.

For the canonical form, $S=m-m^4$ and

$$T=1-20m^3-8m^6.$$

$$\therefore\ T^2+64S^3=(1-20m^3-8m^6)^2+64(m-m^4)^3$$

$$=(1+8m^3)^3, \text{ which is the discriminant.}$$

The canonical form of a nodal cubic is—

$$x^3+y^3+6mxyz=0 \text{ and } S=-m^4,\ T=-8m^6.$$

$\therefore$ T^2+64S^3 is identically zero, as it should be.

256. In § 197 we have shown that the anharmonic ratio σ of a pencil of four tangents drawn from any point on the cubic is related to the parameter m by the equation (4), namely

$$\frac{16m^3(m^3-1)^3}{(8m^6+20m^3-1)^2}=\frac{(\sigma^2-\sigma+1)^2}{(\sigma+1)^2(\sigma-2)^2(2\sigma-1)^2}$$

Thus, this relation becomes

$$\frac{16(-S)^3}{(-T)^2}=\frac{-16S^3}{T^2}=\frac{(\sigma^2-\sigma+1)^2}{(\sigma+1)^2(\sigma+2)^2(2\sigma-1)^2}\cdots\ (1)$$

It follows therefore that when $S=0$, $\sigma^2-\sigma+1=0$; and when $T=0$, $(\sigma+1)^2(\sigma-2)^2(2\sigma-1)^2=0$.

We have seen that in the former case the cubic is equianharmonic and in the latter case harmonic. Hence, the invariant S vanishes for all equianharmonic cubics, and consequently by § 252, the Hessian of all equianharmonic cubics reduce to three right lines and the inflexional tangents intersect three by there in the double points of the Hessian. For all harmonic cubics the invariant T vanishes, and therefore the Hessian of the Hessian *i.e.* the second Hessian coincides with the curve.

257. Covariants of Cubics :

Since the equation of a cubic contains ten coefficients and three variables, altogether making up thirteen, and the general scheme of linear transformation contains nine constants, it follows that the number of independent covariants and invariants must be four, which together with the cubic itself make five. But we have already obtained two invariants S and T, the cubic and its Hessian. Therefore, there must be one other independent covariant of the cubic. Again a covariant of a cubic must be of degree 3, or a multiple thereof, and also a covariant of the canonical form is a linear function of $x^3+y^3+z^3$ and xyz, and consequently of f and H. Therefore, the cubic cannot have any other covariant of degree 3 besides the Hessian. Hence the next covariant must be of degree 6.

Dr. Salmon has given the following covariant* of the sixth order of a cubic :—

Let (x', y', z') be a point and $ax^2+by^2+cz^2+\ldots\ldots=0$ and $a'x^2+b'y^2+c'z^2+\ldots\ldots\ldots=0$ be the polar conics of

* Salmon—H. P. Curves, § 231.

(x', y', z') with regard to the cubic and its Hessian respectively. Then the **F**-conic of these two is

$(BC'+B'C-2FF')x^2+\ldots\ldots\ldots=0$ and is a covariant of the two conics. If it passes through the point (x', y', z'), the locus of (x', y', z') is a covariant of the cubic, which is of degree six in the variables and of eight in the co-efficients.

Thus, the locus of a point such that the **F**-conic of its polar conics with respect to the cubic and its Hessian passes through the point is a sextic covariant of the cubic. The actual expression of this covariant for the general equation has not been calculated. We proceed to find it for the canonical form :—

$$\text{Let } U \equiv x^3+y^3+z^3+6mxyz=0 \quad \ldots \quad \ldots \quad (1)$$

$$\therefore \quad H \equiv m^2(x^3+y^3+z^3)-(1+2m^3)xyz=0 \quad \ldots \quad (2)$$

Then, the polar conic of a point (x', y', z') with regard to (1) is—

$$x'(x^2+2myz)+y'(y^2+2mzx)+z'(z^2+2mxy)=0,$$

$$\text{or } x'x^2+y'y^2+z'z^2+2mx'yz+2my'zx+2mz'xy=0 \quad (3)$$

and the polar conic with regard to (2) is—

$$x'\{3m^2x^2-(1+2m^3)yz\}+y'\{3m^2y^2-(1+2m^3)zx\}$$
$$+z'\{3m^2z^2-(1+2m^3)xy\}=0;$$

$$\text{or, } 3m^2(x'x^2+y'y^2+z'z^2)$$
$$-(1+2m^3)(x'yz+y'zx+z'xy)=0. \quad \ldots \quad (4)$$

Then, after calculating the F-conic of (3) and (4), and then introducing the condition that this **F**-conic passes

* Salmon—Conic Sections § 378.

through the point (x', y', z'), we finally obtain the the locus of (x', y', z') as a sextic covariant Θ of the cubic. Thus the third fundamental covariant Θ of a cubic is—

$$\Theta = 3m^2(1+2m^3)(x^3+y^3+z^3)^2 - m(1-20m^3-8m^6)(x^3+y^3+z^3)xyz - 3m^2(1-20m^3-8m^6)x^2y^2z^2 - (1+8m^3)^2(y^3z^3+z^3x^3+x^3y^3)$$

258. There are two other covariants of the sixth order of a cubic, any one of which could, with equal justice, be selected as the fundamental sextic covariant. In fact, any one of these being selected as the fundamental covariant, the other two are expressible in terms of this. Θ, as we have given above, has been taken by Dr. Salmon as the fundamental covariant, while Mr. Elliot takes $\phi \equiv -(\Theta+3\,\mathrm{USH})$ as the fundamental one.

The other two sextic covariants may be defined as follow:—

(1). The locus of a point whose polar line with regard to the Hessian touches the polar conic of the same point with regard to the cubic, is a covariant of the cubic.

This covariant is found, by the method of § 381, *Ex. 1*, Salmon's Conics, to be $-4(\Theta+3\,\mathrm{SUH})$.

(2) The locus of a point whose polar line with respect to the cubic touches the polar conic of the same point with respect to the Hessian is a sextic covariant of the cubic.

This, after calculation, is found to be

$$-(\mathrm{TU}^2-12\mathrm{SUH}+4\Theta)$$

Thus we see that there are three fundamental covariants of a cubic U, H and Θ. All other covariants are expressible in terms of these three.

There is another irreducible covariant of a cubic which was obtained by Brioschi. This covariant for the semi-canonised form is

$$(abc+8m^3)^3(by^3-cz^3)(cz^3-ax^3)(ax^3-by^3).$$

259. It has been shown that the Hessian of the cubic $x^3+y^3+z^3+6mxyz=0$ is of the same form with a different value of the parameter m, and hence is a cubic belonging to the system having the same nine points of inflexion. It follows therefore that the Hessian of the Hessian is also a cubic of the same system, having analogous properties. In fact, any equation of the form $U+\lambda H=0$ can be expressed in the form $A(x^3+y^3+z^3)+Bxyz=0$. We shall investigate the condition when the Hessian of the Hessian *i.e.* the second Hessian coincides with the original cubic.

The Hessian $H=-m^2(x^3+y^3+z^3)+(1+2m^3)xyz$

$$=x^3+y^3+z^3+6m'xyz=0 \text{ (say)}$$

$$\text{where } 6m'=-\frac{1+2m^3}{m^2} \quad \ldots \quad (1)$$

Therefore, the second Hessian

$$H'=-m'^2(x^3+y^3+z^3)+(1+2m'^3)xyz=0$$

$$=\frac{1}{108\,m^6}\left[(3m^2+12m^5+12m^9)(x^3+y^3+z^3)\right.$$

$$\left.+(1+6m^3-96m^6+8m^9)xyz\right].$$

$\therefore$ The equation of the second Hessian is :—

$$H'\equiv(3m^2+12m^5+12m^9)(x^3+y^3+z^3)$$

$$+(1+6m^3-96m^6+8m^9)xyz=0$$

Now, $H'=\{(4m^2+4m^8-8m^5)$

$$-(m^2-20m^5-8m^8)\}(x^3+y^3+z^3)$$

$$+\{6m(4m^2+4m^8-8m^5)+(1-20m^3-8m^6)(1+2m^3)\}xyz$$

$$=4(m-m^4)^2\{x^3+y^3+z^3+6mxyz\}$$
$$+(1-20m^3-8m^6)\{-m^2(x^3+y^3+z^3)+(1+2m^3)xyz\}$$
$$=4(m-m^4)^2\text{U}+(1-20m^3-8m^6)\text{H}$$
$$=4\text{S}^2\text{U}+\text{TH}. \quad \ldots \quad \ldots \quad (2)$$

From this it follows that if T = 0, the second Hessian coincides with the original curve U.

Hence, the vanishing of the second invariant T of a cubic expresses the fact that the second Hessian coincides with the original curve.

260. Contravariants of Cubics :

All contravariants of a cubic can be expressed in terms of three fundamental contravariants. These are the evectants of S and T, and the reciprocal of the cubic. The two evectants are denoted by P and Q respectively, and the reciprocal by F.

Now, the first evectant of S for the semi-canonical form is $P'=\left(\lambda^3\frac{d}{da}+\mu^3\frac{d}{db}+\nu^3\frac{d}{dc}+\lambda\mu\nu\frac{d}{dm}\right)S.$

Thus, $P=m\ (bc\lambda^3+ca\mu^3+ab\nu^3)+(abc-4m^3)\lambda\mu\nu.$

$\therefore$ For the fully canonised form this contravariant becomes

$$P=m(\lambda^3+\mu^3+\nu^3)+(1-4m^3)\lambda\mu\nu=0 \quad \ldots \quad (1)$$

This is the Cayleyan of the cubic, as we have already shown.

Prof. Cayley calls it the *Pippian*.

The evectant of T for the semi-canonised form is

$$Q'=\left(\lambda^3\frac{d}{da}+\mu^3\frac{d}{db}+\nu^3\frac{d}{dc}+\lambda\mu\nu\frac{d}{dm}\right)T$$

$$=(abc-10m^3)(bc\lambda^3+ca\mu^3+ab\nu^3)-m^2(30abc+24m^3)\lambda\mu\nu.$$

$\therefore$ For the canonical form it becomes—

$$Q=(1-10m^3)(\lambda^3+\mu^3+\nu^3)-m^2(30+24\,m^3)\lambda\mu\nu \quad \ldots \quad (2)$$

This is called by Prof. Cayley the *Quippian* of the cubic. No satisfactory geometrical interpretation has been given to this contravariant. One has been given by Prof. Cayley—in his " A memoir on the curves of the third order"—Coll. Papers Vol. II, p. 396.

The third contravariant F is the reciprocal of the given cubic. Geometrically its vanishing is the condition that the line $\lambda x+\mu y+\nu z=0$ touches the curve, *i.e.* $F=0$ is the tangential equation of the cubic.

For the semi-canonical form it is given by

$$F^1 \equiv b^2c^2\lambda^6+c^2a^2\mu^6+a^2b^2\nu^6$$
$$-(2abc+32m^3)(a\mu^3\nu^3+b\nu^3\lambda^3+c\lambda^3\mu^3)$$
$$-24m^2\lambda\mu\nu(bc\lambda^3+ca\mu^3+ab\nu^3)-24m(abc+2m^3)\lambda^2\mu^2\nu^2;$$

and for the canonical form

$$F=\lambda^6+\mu^6+\nu^6-(2+32\,m^3)(\mu^3\nu^3+\nu^3\lambda^3+\lambda^3\mu^3)$$
$$-24m^2\lambda\mu\nu\,(\lambda^3+\mu^3+\nu^3)-24\,m(1+2m^3)\lambda^2\mu^2\nu^2 \quad \ldots \quad (3)$$

Thus, the three independent contravariants of a cubic are P, Q and F, and are given by (1), (2) and (3).

There is another irreducible contravariant, which is not a rational and integral function of these three. This was discovered by Hermite. For the semi-canonical form it is

$(abc+8m^3)^3(c\mu^3-b\nu^3)(a\nu^3-c\lambda^3)(b\lambda^3-a\mu^3)$; and for the canonical form it is

$$(1+8m^3)^3\,(\mu^3-\nu^3)\,(\nu^3-\lambda^3)\,(\lambda^3-\mu^3).$$

CHAPTER XVI

CURVES OF THE FOURTH ORDER—QUARTIC CURVES.

261. The most general equation of a quartic curve can be written as :—

$$ax^4+4bx^3y+6cx^2y^2+4dxy^3+ey^4+fx^3+3gx^2y+3hxy^2+iy^3$$
$$+lx^2+2mxy+ny^2+px+qy+r=0 \quad \ldots \quad \ldots \quad (1)$$

or in the symbollic form, $u_4+u_3+u_2+u_1+u_0=0 \ \ldots \quad (2)$

Thus the general equation contains 15 arbitrary constants and therefore 14 disposable constants, so that the curve can be made to pass through 14 points chosen arbitrarily, or in other words, fourteen points in general determine a curve of the fourth order uniquely.

262. The most important question which has been solved in different ways by different geometers is the classification of quartic curves. During the first half of the eighteenth century, this was regarded as a very complicated problem by many workers on geometry, among whom the Abbot Bragelogne was the foremost. In two long papers,* he gave a method of determining the different types in which all curves of the fourth order can be groupped. The method is based on the fact that, in general, an equation of the fourth degree in rectangular system of Cartesian co-ordinates can be expressed in canonical forms. But his work throws no light upon the subject.

After Tod Bragelogne, two other well-known geometers, L. Euler and G. Cramer, took up the problem. Both of

* Examen des lignes du quatriéme ordre ou courbes du troisiéme genre. Mém. Acad. Sciences, Paris (1730). Also the Note sur les lignes du quatrième order (Ibid.)

them based the classification of quartics on the nature of the curve at infinity, and its behaviour with the line at infinity. Accordingly, quartic curves can be divided into the nine following classes according to the nature of the four points at infinity on the curve:—(1) All four imaginary in conjugate pairs; (2) Two real and distinct, and two conjugate imaginaries; (3) Four real and distinct; (4) Two conjugate imaginaries and two real and coincident; (5) Two real and distinct, and two real and coincident; (6) Two real double points; (7) Two conjugate imaginary double points; (8) One single and one triple point; (9) A quadruple point.

Each of these classes may again be subdivided into many other forms. Euler and Cramer have given a considerable number of these curves.

263. There is a principle of classification and accordingly curves can be groupped in different ways on different principles. Zeuthen* takes as the basis of his classification of quartic curves the real bitangents of the curve, and thus he has given a complete list of the possible forms of non-singular quartics. Zeuthen first shows that a conic can be described through the points of contact of four bitangents and therefore the equation of a quartic can be put into the form $xyzu=V^2$, where x,y,z,u are the bitangents and V is the conic through their points of contact. His analysis of the possible forms of quartics is made by discussing the different positions of the points of intersection of the four lines with the conic with respect to the quadrilateral formed by them. Thus, when the conic V meets all the lines in real points, Zeuthen divides the quartics into *nine* groups and thirty-six species in all; all other possible cases being easily deducible from these.

* H. G. Zeuthen—" Sur les différentes formes des courbes planes du quatrième ordre. Math. Ann. Bd. VII, p. 411.

264. One of the most important classifications of quartics is based on deficiency ; and they can therefore be groupped in four broad divisions according as the deficiency is 3,2,1,0. The first group contains no singular points, the second contains one double point (a node or a cusp), the third contains two double points and the last has three double points. The following table shows the ten different species, with their corresponding Plücker's numbers, into which the quartic curves can be divided :—

Def. $p=$	Class $m=$	Nodes $\delta=$	Cusps $\kappa=$	Bitang. $\tau=$	Pts. of inflexion $i=$	Degree $n=4$	
3	12	0	0	28	24	4	I
2	10	1	0	16	18	4	II
2	9	0	1	10	16	4	III
1	8	2	0	8	12	4	IV
1	7	1	1	4	10	4	V
1	6	0	2	1	8	4	VI
0	6	3	0	4	6	4	VII
0	5	2	1	2	4	4	VIII
0	4	1	2	1	2	4	IX
0	3	0	3	1	0	4	X.

From the above table it is seen that in each of the last four cases the curve is unicursal. In species IX, the curve is also of the fourth class. In this case, therefore, properties of one quartic can be obtained from another by reciprocation. In the last case, a tri-cuspidal quartic

is of the third class and therefore its properties can be deduced from those of a nodal cubic by reciprocation.

265. The equation of a quartic curve passing through the vertex A of the triangle of reference is

$$x^3 u_1 + x^2 u_2 + x u_3 + u_4 = 0 \quad \ldots \quad (1),$$

where $u_1 = 0$ is the tangent at A. If the point A is a double point, the equation becomes $x^2 u_2 + x u_3 + u_4 = 0 \ldots (2)$, $u_2 = 0$ is the equation of the tangents at the double point. If A is a point of inflexion, u_2 contains u_1 as a factor, and the equation becomes $u_1 x^3 + u_1 v_1 x^2 + u_3 x + u_4 = 0 \quad \ldots \quad (3)$

The equation of a quartic curve passing through the vertices of the triangle of reference is of the form—

$$x^3(ly+mz)+y^3(l'x+m'z)+z^3(l''x+m''y)+fy^2z^2+gx^2z^2$$
$$+hx^2y^2+xyz(px+qy+rz)=0. \quad \ldots \quad \ldots \quad (4)$$

If the vertices are double points on the curve, the co-efficients of x^3, y^3 and z^3 should be absent from the equation, and therefore the equation of a trinodal quartic having the nodes at A, B, C is—

$$fy^2z^2+gz^2x^2+hx^2y^2+xyz(px+qy+rz)=0\;;$$

$$i.e. \quad \frac{f}{x^2}+\frac{g}{y^2}+\frac{h}{z^2}+\frac{p}{yz}+\frac{q}{zx}+\frac{r}{xy}=0. \quad \ldots \quad (5)$$

266. *Any line drawn through a double point on a quartic is harmonically divided by the curve and the first polar of the double point.*

We have shown that a double point on a curve is also a double point, with the same tangents, on the first polar of the same point. Hence, any line drawn through this double point meets the quartic in two other points and the first polar only in one other point.

The equation of a quartic having a double point at the origin O is $u_2+u_3+u_4=0$ (1) and the equation of the first polar is $2u_2+u_3=0. \quad \ldots \quad \ldots \quad \ldots \quad (2)$

Transforming to polar co-ordinates, the equations (1) and (2) become—

$$r^2(a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta)$$
$$+r^3(a'\cos^3\theta+3b'\cos^2\theta\sin\theta+3c'\cos\theta\sin^2\theta+d'\sin^3\theta)$$
$$+r^4(a''\cos^4\theta+\ldots\ldots)=0 \qquad \ldots \qquad \ldots \quad (1)$$

and $2r^2(a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta)+r^3(a'\cos^3\theta+3b'\cos^2\theta\sin\theta+3c'\cos\theta\sin^2\theta+d'\sin^3\theta)=0. \quad \ldots \quad (2)$

Now, if P and Q be the points in which a radius vector meets the quartic, and R be the point where it meets the first polar, we have,

$$\frac{\mathrm{OP}+\mathrm{OQ}}{\mathrm{OP}.\mathrm{OQ}}=\frac{1}{\mathrm{OP}}+\frac{1}{\mathrm{OQ}}$$

$$=\frac{-(a'\cos^3\theta+3b'\cos^2\theta\sin\theta+3c'\cos\theta\sin^2\theta+d'\sin^3\theta)}{a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta}$$

$$=\frac{2}{\mathrm{OR}} \quad \text{[by equation (2)]}$$

$\therefore \quad \dfrac{1}{\mathrm{OP}}+\dfrac{1}{\mathrm{OQ}}=\dfrac{2}{\mathrm{OR}}$, *i.e.* (OPRQ) is a harmonic range.

267. A quartic curve may be defined as the *locus of intersections of two homographic pencils of conics.*

[This is analogous to the definition of a conic as the locus of intersections of corresponding rays of two homographic pencils of lines].

Let the two pencils of conics be defined by

$\mathrm{P}+\lambda\mathrm{Q}=0$ (1) and $\mathrm{P}'+\mu\mathrm{Q}'=0$ (2) respectively, where λ and μ are parameters; and P, Q; P′, Q′ are the pairs of base-conics of the two pencils. If the two pencils are homographic, *i.e.* if there is a (1, 1) correspondence between the members of the pencils, λ and μ must be connected by the homographic relation of the form

$$\mathrm{A}\lambda\mu+\mathrm{B}\lambda+\mathrm{C}\mu+\mathrm{D}=0 \qquad \ldots \quad (3)$$

If therefore we eliminate λ and μ between (1), (2) and (3), we obtain

$$A.\frac{PP'}{QQ'}-B.\frac{P}{Q}-C\frac{P'}{Q'}+D=0,$$

$$i.e.\ APP'-BPQ'-CP'Q+DQQ'=0, \qquad \ldots \quad (4)$$

which, therefore, represents the locus of intersections of the corresponding members of two homographic pencils. The equation (4) is of the fourth degree and therefore represents a quartic curve.

From the form of the equation it follows that this curve passes through the four base-points of intersection of P and Q, as also through the four base-points of intersection of P′ and Q′. Thus the above quartic curve passes through the eight given fixed points.

268. It has been shown (Salmon's Conics § 259) that the anharmonic ratio of a pencil of lines, which join four fixed points on a conic to a variable point, is constant and is a function of the mutual distances of the four points and the constants in the equation of the conic. It follows, therefore, that in a pencil of conics $P+\lambda Q$ (which pass through four fixed points) each individual member is characterised by a definite value of this anharmonic ratio σ, which is a linear function of the parameter λ. For any particular member we have, therefore, $\sigma=k\lambda$, where the value of k depends upon the position of the four base points. Similarly, for the conics of a second pencil $P'+\mu Q'$, we have $\sigma'=k'\mu$, where the value of k' depends upon the four fixed base points.

If the two pencils are homographic,

$$A\lambda\mu+B\lambda+C\mu+D=0.$$

$$\therefore A.\frac{\sigma\sigma'}{kk'}+B\frac{\sigma}{k}+C\frac{\sigma'}{k'}+D=0,$$

i.e. $$A\sigma\sigma'+Bk'\sigma+Ck\sigma'+Dkk'=0.$$

Hence, since k and k' depend upon the base points and are consequently constants, the above relation may be written as

$$a\sigma\sigma'+b\sigma+c\sigma'+d=0 \quad \ldots \quad \ldots \quad (5)$$

Thus, if T is a point of intersection of two corresponding members of the two pencils, whose base-points are P_1, Q_1, R_1, S_1 and P_2, Q_2, R_2, S_2 respectively, then the anharmonic ratios of the two pencils T $(P_1Q_1R_1S_1)$ and T $(P_2Q_2R_2S_2)$ are connected by the relation (5), and the locus of T is a quartic curve through these base-points. Therefore, we may define the quartic curve as follows :—The locus of a point T which moves such that the cross-ratios of the pencils T $(P_1Q_1R_1S_1)$ and T $(P_2Q_2R_2S_2)$ are connected by a homographic relation, is a quartic curve through the eight points.

Now, the points T, P_1, Q_1, R_1, S_1 lie on the quartic. Therefore the conic through these five points intersect the quartic in three other points U,V,W, which are also points on the corresponding member of the second pencil. The three points U,V,W possess properties analogous to those of T. Thus the two corresponding conics through P_1,Q_1,R_1,S_1 and P_2,Q_2,R_2,S_2 meet the quartic in four common points T,U,V,W. Hence, this latter group of points is residual to each of the given base-groups. Therefore the two groups of base-points are co-residual groups on the quartic.

269. Thus we see that the two groups of base-points are not arbitrary, but they are co-residual groups. Hence in constructing the quartic we proceed as follows :—

Take P_1, Q_1, R_1, S_1 any four arbitrary points on the quartic. Through these four points describe a conic, intersecting the quartic in four other points T,U,V,W. Through these latter points and an assumed point P_2, describe another conic, which will intersect the quartic in three other points Q_2, R_2, S_2. Thus the group

$P_2Q_2R_2S_2$ is co-residual to $P_1Q_1R_1S_1$, and they may be taken as the base-points.

270. Let us consider the case when the base-point P_1 coincides with P_2. Then, two corresponding members of the pencils always intersect at the point P_1 (P_2), and consequently in three other points only. The tangents at P_1 (P_2) to the corresponding conics form two homographic pencils of lines, and therefore they have two self-corresponding rays.* Thus there are two pairs of corresponding conics which have a common tangent at P_1 (P_2). Now, since the intersections of two corresponding conics lie on the quartic, therefore the common tangent to a pair of corresponding conics is also a tangent to the quartic at P_1 (P_2). Hence the quartic has two tangents at the point P_1 (P_2) which, therefore, is a double point on the curve. Similarly, it can be shown that if Q_1 and Q_2 coincide, Q_1 (Q_2) is a double point on the curve. If further R_1 (R_2) coincide, it is also a double point.

Hence a trinodal quartic may be defined as follows :—
The locus of intersection of two homographic pencils of conics, which have three fixed base-points common, is a quartic having double points at the three common points.

271. We have seen in §264 that quartics can be divided into ten species, according to the nature and number of double points they possess. But there are other special forms, which possess complex singularities, arising from the union of two or more of the double points. We give below some cases of these complex singularities :—

(1) **A tacnode :** Two nodes may coincide as consecutive points on a curve, giving rise to the singularity

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* Scott—loc. cit. § 164.

called a *tacnode.* The tangent at a tacnode has a contact of the third order with the curve and therefore cannot meet the curve in any other point. In fact, two branches of the curve have a simple contact at a tacnode. The tacnodal tangent counts as two double tangents.

The general equation of a quartic having a tacnode at the origin, with the axis of x as the tacnodal tangent, can be written as $y^2+bx^2y+cxy^2+dy^3+ex^4+fx^3y+gx^2y^2$

$$+hxy^3+iy^4=0 \quad \ldots \quad (1)$$

For, the shape of the curve near the origin is given by $y^2+bx^2y+ex^4=0$ (§112) which gives

$$y=x^2\left(\frac{-b\pm\sqrt{b^2-4e}}{2}\right)$$

i.e. $y=\lambda x^2$, and $y=\mu x^2$, where λ and μ are the roots of

$$\rho^2+b\rho+e=0. \quad \ldots \quad \ldots \quad (2)$$

Hence, there are two branches of the curve whose forms near the origin are the same as those of the parabolas $y=\lambda x^2$, and $y=\mu x^2$, and consequently the two branches touch each other at the origin, which is therefore a tacnode, with $y=0$ as the tacnodal tangent.

(2) **A Rhamphoid cusp:** This is formed by the coincidence of an ordinary cusp and a node. This has been called so, from a fancied resemblance to the form of a beak. The tangent at such a point counts once as a double tangent and once as a stationary tangent.

The general equation of a quartic having a rhamphoid cusp at the origin is—

$$(y-\lambda x^2)^2+cxy^2+dy^3+fx^3y+gx^2y^2+hxy^3+iy^4=0 \ldots \quad (3)$$

This is a particular case of (1), when the roots of the equation (2) are equal, *i.e.* when $\lambda=\mu$.

The shape of the curve near the origin is given by

$$y=\lambda x^2\pm kx^{\frac{5}{2}} \text{ or } (y-\lambda x^2)^2-k^2x^5=0.$$

(3) **An oscnode:** This is formed by the union of three nodes as consecutive points on the curve, and is

called an *oscnode.* At an oscnode two branches of the curve have a three-pointic contact.

The general equation of a curve having an oscnode at the origin is—

$$(y-\lambda x^2)^2+cxy(y-\lambda x^2)+dy^3+gx^2y^2+hxy^3+iy^4=0 \quad .. \ (4)$$

The forms of the two branches of the curve near the origin are given by $y=\lambda x^2+k_1x^3$ and $y=\lambda x^2+k_2x^3$, where k_1 and k_2 are the roots of $k^2+c\lambda k+\lambda^2(d\lambda+g)=0$.

These show that the two branches cross as well as touch each other at the origin. The oscnodal tangent counts as three double tangents, and there is only one other ordinary double tangent.

(4) **A Tacnode cusp :** It is formed by the union of two nodes and a cusp, or a tacnode and a cusp, as consecutive points on a curve. The general equation of a curve having a tacnode cusp at the origin is of the form

$$(y-\lambda x^2-cxy-dy^2)^2=\mathrm{A}xy^3+\mathrm{B}y^4.$$

(5) Three double points may coincide at a point giving rise to a *triple point.* The general equation of a quartic having a triple point at the origin is of the form $u_3+u_4=0$, the three tangents at the origin being given by $u_3=0$. This is a cubic in y/x. Four distinct cases are to be considered, according as the tangents are (*i*) all real and distinct, (*ii*) one real and distinct and two real and coincident, (*iii*) all real and coincident, (*iv*) one real and two imaginary. Hence there are four species of triple points.

272. There are other special kinds of complex singularities which deserve special consideration. A node may be a point of inflexion on one or both branches of the curve passing through it. The two cases are distinguished as follow :—

(1) **A flecnode** It is a node, which is a point of inflexion on one branch of the curve and consequently

one of the tangents at which is a stationary tangent. Such a point may be considered as arising from the union of an ordinary node with a point of inflexion. Every flecnodal tangent has a four-pointic contact with the curve, since it has a three-pointic contact with the branch it touches and cuts the other branch. The equation of a quartic having a flecnode at the origin is $u_1 v_1 + u_1 v_2 + u_4 = 0$, where u_1, v_1, v_2, u_4 are expressions in x and y. $u_1 = 0$ is the equation of the flecnodal tangent. The reciprocal polar of a flecnode is a double tangent, which has a simple contact at one point with the reciprocal curve and touches it at a cusp at the other.

(2) **A biflecnode:** It is a node at which both the tangents are stationary tangents. Such a point may be regarded as arising from the union of two points of inflexion with a node. A biflecnode has properties analogous to those of points of inflexion on a cubic. The general equation of a quartic having a biflecnode at the origin is $u_2 v_0 + u_2 v_1 + u_4 = 0$. The reciprocal polar of a biflecnode is a pair of cusps having a cuspidal tangent.

273. There is another kind of singular point which a quartic curve can possess. *A point of undulation* is defined as a point where the tangent has a contact of the third order with the curve. In fact, as we shall see later, a point of undulation is formed by the union of two points of inflexion as consecutive points on a curve. The tangent at such a point is equivalent to two inflexional tangents and one ordinary double tangent. It will be shown in § 284. that the equation of a quartic can be put into the form $\alpha . \beta . \gamma . \delta = S^2$, where α, β, γ, δ touch the quartic at the two points where each intersects the conic S. If now α, β, γ, δ are tangents to S, then α, β, γ, δ are each a tangent to the quartic which has a contact of the third order, *i.e.*

these points of contact will be points of undulation. Thus a quartic has only four real points of undulation.

274. When α, β, γ, δ are tangents at points of undulation on a quartic, the conic S is inscribed in the quadrilateral. Four triangles can be formed by taking any three of these four tangents, and any of these triangles can be taken as the triangle of reference, of which S is the inscribed conic But it is proved in Conics * that the three lines joining the vertices of any triangle circumscribing a conic to the points of contact of the opposite sides meet in a point. Hence we obtain the theorem :—

If a triangle is formed by the tangents at any three real points of undulation, the lines joining the vertices of the triangle with the points of contact of the opposite sides meet in a point.

275. *If a quartic has three points of undulation lying on a right line, the fourth point where the line meets the curve again is a point of undulation.*

This is easily proved by the principle of residuation :—

Let A, B, C, be the three collinear points of undulation on the quartic, and let D be the fourth point where ABC meets the curve again.

Now, since each of A, B, C is a point of undulation, we have—

$$[4A]=0, \qquad [4B]=0, \qquad [4C]=0.$$

$$\therefore \qquad [4A+4B+4C]=0.$$

$$\text{But} \qquad [A+B+C+D]=0.$$

$$\therefore \qquad [4A+4B+4C+4D]=0.$$

$$\therefore \qquad [4D]=0, \textit{ i.e. } D \text{ is a point of undulation.}$$

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* Salmon—Conics § 129.

276. We shall now prove the following theorem, which is only a particular case of a more general theorem on polar curves :—

The locus of points, whose first polars break up into a conic and a right line, passes through every point, where the tangent has a contact of the third order with the quartic.

The equation of a quartic which passes through the origin is $u_1+u_2+u_3+u_4=0$, where u_1 is the tangent at the origin. If this tangent has a contact of the third order with the curve, u_1 must be a factor in each of u_2 and u_3; consequently the equation of the curve becomes $u_1(v_0+v_1+v_2)+u_4=0$.

Now, by § 54 is the first polar of the origin—

$u_1(3v_0+2v_1+v_2)=0$, which evidently breaks up into the tangent u_1 and a conic. Hence the theorem.

Note: We have seen that the first polar of any point passes through the double points on the curve. Hence the first polar of a flecnode breaks up into a line and a conic.

277. *A quartic curve cannot have more than two flecnodes.*

The equation of a quartic having three nodes at the vertices A, B, C of the fundamental triangle is

$$ay^2z^2+bz^2x^2+cx^2y^2+xyz(lx+my+nz)=0.$$

If A is a flecnode, the coefficients of x^2 and x should have a common linear factor. This requires that $yz(my+nz)$ should contain with $lyz+bz^2+cy^2$ a common linear factor; *i.e.*

$$lyz+bz^2+cy^2\equiv(my+nz)(m'y+n'z)$$

$$\therefore \quad mm'=c, \quad nn'=b, \quad \text{and} \quad mn'+m'n=l;$$

or $$bm^2+cn^2=lmn \qquad \dots \quad (1)$$

Similarly, if B is a flecnode, we must have

$$al^2+cn^2=lmn \qquad \dots \quad (2)$$

If now the third vertex C is a flecnode we must have in addition $al^2+bm^2=lmn$. ... (3)
Thus from (1), (2), (3) we obtain

$$al^2+bm^2=bm^2+cn^2=cn^2+al^2=lmn, \text{ which give}$$
$$al^2=bm^2=cn^2=\tfrac{1}{2}lmn=4abc.$$

But these values of the constants reduce the equation to a perfect square, and therefore the third vertex can not be a flecnode.

278. *If a trinodal quartic has two biflecnodes, the third node must also be a biflecnode.*

The equation of a quartic having three nodes at A, B, C is

$$ay^2z^2+bz^2x^2+cx^2y^2+xyz(lx+my+nz)=0.$$

If A is a biflecnode, the coefficient of x should contain the coefficient of x^2 as a factor *i.e.* $yz(my+nz)$ should contain bz^2+cy^2+lyz as a factor. But this requires that third powers of y or z should occur, which is impossible, since B and C are also nodes on the curve. Hence the only possible conclusion is that $l=m=n=0$; and thus the coefficient of x vanishes. Hence the equation of a quartic having biflecnodes at A and B is—

$$\frac{a}{x^2}+\frac{b}{y^2}+\frac{c}{z^2}=0.$$

The symmetry in the result shows that the third vertex C is also a biflecnode.

In order that the curve may be real, one of the constants must have a negative sign (say c). Hence it follows that the nodal tangents at A and B are real, while those at C are imaginary.

279. A biflecnode on a quartic possesses harmonic properties analogous to those possessed by a point of inflexion on a cubic. The equation of a quartic having a biflecnode at the origin O is $u_2v_0+u_2v_1+u_4=0$. (1)

The first polar of O is $u_2(2v_0+v_1)=0$. (2) Hence the first polar of a biflecnode consists of the two biflecnodal tangents and another line which is called the *harmonic polar* of the biflecnode.

Consider a line drawn through the biflecnode at the origin. Let $u_2=ax^2+2hxy+by^2, v_1=lx+my$; $u_4=a'x^4+4b'x^3y+6c'x^2y^2+4d'xy^3+e'y^4$.

Transforming to polar co-ordinates, the equation (1) becomes

$$r^2(a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta)(v_0+rl\cos\theta+rm\sin\theta)$$
$$+r^4(a'\cos^4\theta+\ldots\ldots)=0.$$

If P and Q be the points in which the radius vector intertects the curve again, then

$$\frac{OP+OQ}{OP.OQ}=\frac{1}{OP}+\frac{1}{OQ}=\frac{-(l\cos\theta+m\sin\theta)}{v_0}$$

If R be the point in which the radius cuts the line $2v_0+v_1=0$, we have

$$\frac{1}{OR}=\frac{-(l\cos\theta+m\sin\theta)}{2v_0}$$

$$\therefore\ \frac{1}{OP}+\frac{1}{OQ}=-\frac{l\cos\theta+m\sin\theta}{v_0}=\frac{2}{OR}$$

which shows that OR is the harmonic mean between OP and OQ. Hence we obtain the theorem :—

Every line drawn through a biflecnode on a quartic is divided harmonically by the curve and the harmonic polar.

280. *If two right lines be drawn from a biflecnode to meet a quartic in four points and their extremities be joined directly and transversely, the points of intersection will lie on the harmonic polar.*

This follows immediately from the harmonic properties of a complete quadrilateral. The method of proof is exactly similar to that used in § 174 in the case of a cubic curve.

Cor : In exactly the same manner as in § 176. we deduce the following—*The tangents at the extremities of any chord drawn through a biflecnode intersect on the harmonic polar.*

281. *The harmonic polar passes through every double point on a quartic.*

This follows as a particular case of the theorem in § 279. For, if O is a biflecnode, and P a double point on the quartic, the line OP cannot meet the curve in any other point. But OP is divided harmonically by the curve and the harmonic polar. Hence the point where OP meets the harmonic polar coincides with P. Therefore P lies on the harmonic polar.

Cor : Besides the biflecnode, a quartic may have two other double points.

Hence the line joining these double points is the harmonic polar of the biflecnode.

This may be analytically proved as follows :—The equation of a quartic having a biflecnode at A is—

$$x^2 u_2 v_0 + x u_2 v_1 + u_4 = 0.$$

Or, $(ay^2+2hyz+bz^2)\{v_0 x^2 + x(ly+mz)\} + u_4 = 0 \;\ldots\; (1)$

If B and C are nodes, the equation (1) cannot contain y^4, y^3, z^4, z^3, which requires that $u_4 = \lambda y^2 z^2$, and either $a=b=0$ or $l=m=0$.

But if $a=b=0$, the quartic breaks up into a conic and two lines, which is therefore inadmissible. Hence $l=m=0$; and the equation becomes $v_0 x^2 u_2 + \lambda y^2 z^2 = 0$ and the harmonic polar A is $x=0$, which is the line BC.

282. Let us consider the equation $UW=V^2$, where U, V, W each represents a conic. This equation implicitly contains sixteen constants, but the equation of a quartic contains 14 independent constants and therefore the equation of any quartic can be reduced to this form in a doubly infinite number of ways.

Let $ax^4+by^4+cz^4+\ldots\ldots=0$ be the equation of a quartic (1). If this is to reduce to the form $UW=V^2$, the equation (1) is to be identified with

$$(x^2+b'y^2+c'z^2+\ldots)(x^2+b''y^2+c''z^2+\ldots)$$
$$-(a_1x^2+b_1y^2+c_1z^2+\ldots)^2=0 \quad \ldots \quad (2)$$

which contains 16 constants.

If the expanded form of (2) is

$$Ax^4+By^4+Cz^4+\ldots=0 \quad \ldots \quad (3)$$

comparing the co-efficients of (1) and (3), we have

$$\frac{a}{A}=\frac{b}{B}=\frac{c}{C}=\ldots \quad \ldots \quad \ldots \quad (4)$$

There are 14 conditions in the equations (4). Thus the 16 constants must satisfy 14 conditions and therefore the equation can be reduced in a doubly infinite number of ways.

The form of the equation shows that the conics U and W each intersects the conic $V=0$ in four points lying on the quartic, and each of them cuts a contiguous conic $V'=0$ in four contiguous points on the quartic. Hence the conics U and W each touch the quartic in four points where they respectively meet $V=0$.

283. As in the case of conics, the quartic $UW=V^2$ (1) may be regarded as the envelope of the variable conic $\lambda^2U+2\lambda V+W=0$ (2), where λ is a variable parameter.

The equation (1) represents a family of conics, each individual member being obtained by giving a particular value to λ. A conic consecutive to (2), given by $(\lambda+\delta\lambda)^2 U + 2(\lambda+\delta\lambda)V+W=0$, intersects (2) in four points which are the points of contact with the envelopes. Hence the conic (2) touches the quartic (1), where $\lambda U+V=0$ and consequently $\lambda V+W=0$ *i.e.*, in the four points determined by $\lambda U+V=0$ and $\lambda V+W=0$. Thus every curve of the family (2) has a quadruple contact with the quartic.

The discriminant of the form $\lambda^2 U+2\lambda V+W$ is a function of the third degree in its co-efficients and therefore will involve λ in the sixth degree. Thus, six values of λ can be determined so that this discriminant vanishes and consequently there are six conics of the system (2) which reduce to a pair of right lines. Now each of these pairs of right lines will touch the quartic in four points. But since a line cannot meet a quartic in more than four points, each line of a pair touches the quartic at two distinct points *i.e.*, each line is a bi-tangent of the quartic. Thus there are twelve bi-tangents of the quartic.

284. Now the equation $UW=V^2$ can again be written in the form

$$(\lambda^2 U+2\lambda V+W)(\mu^2 U+2\mu V+W)$$
$$=\{\lambda\mu U+(\lambda+\mu)V+W\}^2 \quad \ldots \quad (3)$$

as can easily be shown by multiplying out both sides of (3), where λ and μ are arbitrary parameters. Now from what has been said in the preceding article it follows that both the conics touch the quartic at the four points, where each meets the conic $\lambda\mu U+(\lambda+\mu)V+W=0$. Hence the two sets of four points at which any two of the enveloping conics touch the quartic lie on another conic.

Again, if we determine λ and μ such that $\lambda^2 U + 2\lambda V + W$ and $\mu^2 U + 2\mu V + W$ each represents a pair of right lines, the equation (3) takes the form

$$\alpha\beta.\gamma\delta = S^2 * \quad \ldots \qquad \ldots \quad (4),$$

where $S = \lambda\mu U + (\lambda + \mu)V + W$, in which the above values of λ and μ have been substituted. From the form of the equation (4) it follows that each of the lines $\alpha, \beta, \gamma, \delta$ touches the quartic at two points, where it meets the conic $S=0$, and therefore is a bi-tangent or double tangent to the quartic. Also the eight points of contact of these four bi-tangents lie on a conic $S=0$. Hence we obtain the theorem :—

The points of contact of any four bi-tangents of a quartic lie on a conic.

Notice that the reduction of the equation to the form (4) can be effected in only $5 \times 6 = 30$ different ways. For λ and μ have each six different values and when we give any value to λ, we must give to μ any one of the five remaining values.

285. Given a pair of bi-tangents touching the quartic at the points A_1, B_1 and A_2, B_2 respectively. If there is another bi-tangent touching the quartic in A_3, B_3, lying on a conic passing through A_1, B_1, A_2, B_2, then the two points A_4, B_4 where this conic cuts the quartic again are the points of contact of a fourth bi-tangent.

This follows immediately from the theorem of the previous article. We may prove this article independently by residuation as follows :—

We have $[2A_1 + 2B_1] = 0$, $[2A_2 + 2B_2] = 0$
and $[2A_3 + 2B_3] = 0$.

* This form is due to Plücker who used it in obtaining the bi-tangents of a quartic—Theorie der Algebraischen Curven.

$\therefore$ By addition $[2A_1+2B_1+2A_2+2B_2+2A_3+2B_3]=0.$

Again, $[A_1+B_1+A_2+B_2+A_3+B_3+A_4+B_4]=0$

or $[2A_1+2B_1+2A_2+2B_2+2A_3+2B_3+2A_4+2B_4]=0.$

$\therefore$ By subtraction $[2A_4+2B_4]=0.$

Hence A_4 and B_4 are the points of contact of a fourth bi-tangent.

If now A_1 and B_1 coincide, the line A_1B_1 meets the quartic in four consecutive points at A_1, which is then called *a point of undulation.* We have seen that if $\alpha\beta\gamma\delta=S^2$ be the equation of a quartic, $\alpha, \beta, \gamma, \delta$ are bi-tangents, their points of contact being the points where each intersects the conic S. Therefore if $\alpha, \beta, \gamma, \delta$ are tangents to S, the points of contact are the points of undulation on the quartic.

286. We have just now shown that the enveloping conics of a quartic and the conics through the points of contact of any pair of them belong to the system included in the equation $lU+mV+nW=0$. But if $lU+mV+nW=0$ represents a pair of right lines, their intersection lies on the Jacobian of U, V, W.* Hence it follows that if the points of contact of an enveloping conic are joined by three pairs of right lines, the intersection of each pair lies on the Jacobian of U, V, W. But this Jacobian is the Hessian of a certain cubic, of which $lU+mV+nW=0$ is the net of polar conics. (§ 200). Hence each of these lines touches the Cayleyan of the cubic.

Again, since the six pairs of bitangents are included in the from $\lambda^2U+2\lambda V+W$, it follows that these bitangents all touch the Cayleyan and the intersection of each pair lies on the Jacobian of the system. It further appears

* Salmon—Conics § 388.

that the lines which join directly or transversely the points of contact of any pair of these bitangents all touch the Cayleyan and they intersect on the Jacobian.

287. Salmon's Theorem.*

Through the four points of contact of any two bitangents, five conics can be described, each of which passes through the four points of contact of two other bitangents.

The equation of a quartic may be written as $UW = V^2$. If now W breaks up into two linear factors α, β, we have $\alpha\beta . U = V^2$. ... (1) and α, β are bitangents to the quartic. This equation can again be written as—

$$\alpha\beta(\lambda^2 U + 2\lambda V + \alpha\beta) = (\alpha\beta + \lambda V)^2 \qquad \ldots \quad (2)$$

Now, the discriminant of $\lambda^2 U + 2\lambda V + \alpha\beta$ is of the sixth degree in λ and this involves λ as a factor. The value $\lambda = 0$ corresponds to the bitangents α and β. The other five values of λ for which the discriminant vanishes will reduce $\lambda^2 U + 2\lambda V + \alpha\beta$ to a pair of right lines. Therefore there are five different ways in which the equation (1) can be reduced to the form $\alpha\beta\gamma\delta = (\alpha\beta + \lambda V)^2$. These conics pass through the points of contact of the four bitangents $\alpha, \beta, \gamma, \delta$. Hence the theorem.

A non-singular quartic has 28 bitangents, and there are therefore $\frac{1}{2}.28.27$ or 378 pairs of bitangents. Each of these pairs give rise to five different conics, but each conic may arise from any one of the six different pairs formed by the four bitangents which correspond to that conic. Hence there are in all $\frac{5}{6} \times 378$ or 315 conics, each of which passes through the points of contact of four bitangents of a quartic.†

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* Salmon—H. P. Curves § 255.

† This investigation is due to Salmon. Other workers have discussed the system of these conics—Cayley—Coll. Papers, Vol. VII, p. 123. Hesse—Crelle's Journal, Vol. XLIX, p. 243.

288. We have seen that $\lambda^2 U+2\lambda V+W$ touches the quartic at the four points determined by $\lambda U+V$, $\lambda V+W$. If, however, $\lambda U+V$ and $\lambda V+W$ touch each other, then the conic $\lambda^2 U+2\lambda V+W$ touches the quartic in two ordinary points and has a four-pointic contact at one point, and this last point is a point on the Jacobian. Hence, at each of the points where the Jacobian intersects the quartic, the enveloping conic has a contact of the third order with the curve and there are therefore 12 such conics.

289. We give below an example of a quartic which has been given by Plücker,* for showing that the 28 bitangents of a quartic are all real. In this case the quartic must necessarily be quadripartite. These four branches have therefore ${}^4C_2\times 4=24$ common tangents, which are bitangents of the quartic. There must be other four bitangents for which Plücker starts with the equation

$$\Omega\equiv(y^2-x^2)(x-1)(x-\tfrac{3}{2})-2\{y^2+x(x-2)\}^2=0.$$

The circle within the parenthesis is circumscribed about the triangle $(y^2-x^2)(x-1)$. Therefore the curve has double points at the three vertices, and $x-\frac{3}{2}=0$ is a double tangent, as is shown in Fig. 35 by the thick line.

Let us now consider the curve $\Omega=k$, for small integral positive or negative values of k.

The curve $\Omega=k$ does not meet $\Omega=0$ in any finite point, and it deviates less from the form of the curve Ω, the less we suppose k, and according as k is positive or negative, it is altogether within or without Ω. When k is negative, the curve is altogether without and it is unipartite and has 28 real bitangents.? When the curve is within, it consists of four ovals, one in each of the compartments into which the curve Ω is divided. Each oval has one

* Plücker—Theorie der Algebraischen Kurven, p. 247.

tangent touching it doubly. Hence there are four bitangents belonging to the ovals. Again, any two ovals have four common tangents and the four ovals can be grouped into six pairs and there are therefore 24 bitangents. Thus altogether there are 28 bitangents.

The bitangents which touch a single branch of the curve are called *bitangents of the first kind*, and the bitangents which touch two different branches are called *bitangents of the second kind*. Thus, in the above example, there are 4 bitangents of the first kind and 24 of the second kind.

290. When a tangent touches a single branch at two real points, it is evident that the arc at each of these points is convex towards the tangent. Therefore, intermediate between these two convex parts, there must be a part of the arc which is concave towards the tangent. This concave part must therefore be separated from the convex parts by a point of inflexion at each extremity. Therefore, corresponding to each bitangent of the first kind, there are two real points of inflexion. Conversely, the existence of two real points of inflexion on a single branch implies the presence of a bitangent of the first kind. Now, since there are only four such bitangents, *a quartic curve cannot have more than eight real points of inflexion.*

If, however, the two points of inflexion coincide, the bitangent touches the curve at four consecutive points and the point of contact becomes a point of undulation. Hence, since there are only eight real points of inflexion, a quartic can have at the most four real points of undulation.

CHAPTER XVII.

TRINODAL QUARTICS.

291. We have seen that the maximum number of double points which a quartic can possess is three, and consequently a trinodal quartic is necessarily unicursal (§ 41).

The equation of a quartic having nodes at the three vertices of the fundamental triangle can be written as :—

$$ay^2z^2+bz^2x^2+cx^2y^2+2fx^2yz+2gy^2zx+2hz^2xy=0 \quad \ldots \quad (1)$$

which can again be written as—

$$\frac{a}{x^2}+\frac{b}{y^2}+\frac{c}{z^2}+2f\frac{1}{yz}+2g\frac{1}{zx}+2h\frac{1}{xy}=0.$$

Comparing this with the equation of the conic

$$aX^2+bY^2+cZ^2+2fYZ+2gZX+2hXY=0, \quad \ldots \quad (2)$$

it is seen that the expressions for X, Y, Z in terms of a single parameter t give the desired expressions of x, y, z by means of the relations

$$x:y:z=\frac{1}{X}:\frac{1}{Y}:\frac{1}{Z}.$$

Now, when X, Y, Z are expressed in terms of a single parameter, $X:Y:Z=f_1(t):f_2(t):f_3(t)$, the parametric representation of the quartic becomes

$$x:y:z=f_2f_3:f_3f_1:f_1f_2.$$

The general method, however, consists in taking a pencil of conics through the three nodes and any fourth point on the quartic. Then, of the eight intersections of

a conic and a quartic, seven are known and consequently there must be a linear relation giving the remaining intersection in terms of the parameter of the pencil.

292. Thus it is seen that the quartic may be generated from a conic by writing in its equation $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ for x, y, z respectively. This process is known as "Quadric Inversion".* This transformation can easily be effected by means of a geometrical construction and was given by Dr. Hirst in the Proceedings of the Royal Society in the year 1865. In this process a point corresponds to a point, but a line in general corresponds to a conic. It is in fact the circular inversion generalised. We take a point O as the origin of inversion and a fixed conic as base. In order to find the inverse of a point P, we have to determine the point P′, where OP intersects the polar line of P with respect to the conic. Thus it follows that to any position of P corresponds a single definite position of P′, and *vice versa.* But there are exceptional positions of P when that of P′ becomes indeterminate. There is no room for a detailed account of the theory in the present volume; for a fuller treatment of the subject, the reader is referred to the chapter on Correspondence by Scott.*

In fact, the properties of a trinodal quartic can be deduced from those of a conic by Quadric Inversion. The relation is more definitely established by deriving the equation of one from that of the other. Thus, if u, w be the tangents to a conic and v their chord of contact, its equation can be written as $uw=v^2$. From this the equation of the quartic can at once be written down in the form $UW=V^2$, where U, V, W are linear functions of yz, zx, xy.

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* C. A. Scott—loc. cit. § 232.

293. Let P and P′ be two inverse points whose co-ordinates are respectively (x, y, z) and (x', y', z').

Then, $$x : y : z = y'z' : z'x' : x'y'.$$

Now, if P be any point on the side BC $(x=0)$, then $y'z'=0$, *i.e.* $y'=0$ and $z'=0$; that is to say, the point P′ coincides with A. Reciprocally, to A corresponds any point on BC. But if P′ is a point on BC, *i.e.* $x'=0$, both y and z vanish, but still they have to each other a definite relation $z' : y'$. Therefore, to any point on BC there corresponds a point indefinitely near to A, but in a definite direction. If $y=\lambda x$ be the equation of AP, then the equation of AP′ is $x'=\lambda y'$, and the two lines make equal angles with the sides AB and AC. Therefore the inverse point P′ is very near to A, but in the direction of the line $x=\lambda y$.

Again, the conic meets each side of the triangle (say BC) in two points. Corresponding to these two points, we have two points indefinitely near to A, but in definite directions, namely, on the tangents to the quartic at the node A. Thus, to the lines joining A to the points on BC where the conic intersects it, there correspond the nodal tangents at A. It should be remarked here that the vertices are nodes, cusps or conjugate points on the quartic, according as the conic intersects the opposite sides in two real and distinct, coincident or conjugate imaginary points. Considering the trinodal quartic as the inverse of a conic, we can deduce a number of properties of the former curve from known properties of the latter.

294. Theorem : *The six nodal tangents of a trinodal quartic touch one and the same conic.**

* The proof given here is taken from Salmon's H. P. Curves § 285.

Consider a conic which intersects the sides of the triangle in six points.

Let the equations of lines joining the vertices to the intersections on opposite sides be respectively

$$y=\lambda z,\ y=\lambda' z\ ;\ z=\mu x,\ z=\mu' x\ ;\ x=\nu y,\ x=\nu' y.$$

Then, by Carnot's Theorem* it follows that

$$\lambda\lambda'\mu\mu'\nu\nu'=1.$$

This relation remains unchanged, when λ, μ, ν are changed into their reciprocals, and consequently the inverse lines meet the sides in six points which also lie on a conic. Again, it can easily be proved that the six lines joining the vertices to the points in which a conic intersects the sides all touch a conic. It follows, therefore, that the six inverse lines all touch a conic; but these inverse lines are the nodal tangents to the trinodal quartic. Therefore the nodal tangents to a trinodal quartic all touch one and the same conic.

We can supply the following analytical proof :—

Let the equation of the quartic having three nodes at A, B, C be—

$$ay^2z^2+bz^2x^2+cx^2y^2+2fx^2yz+2gy^2zx+2hz^2xy=0.$$

Then the three pairs of nodal tangents are—

$$cy^2+bz^2+2fyz=0,\ az^2+cx^2+2gzx=0,\ ay^2+bx^2+2hxy=0.$$

Let $bx^2+ay^2+2hxy=(l_1x+m_1y)(l_2x+m_2y)=0.$

Then, $\dfrac{m_1}{l_1}+\dfrac{m_2}{l_2}=+\dfrac{2h}{b}$ and $\dfrac{m_1}{l_1}\cdot\dfrac{m_2}{l_2}=\dfrac{a}{b}.$

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* L. Cremona—Elements of Projective Geometry, § 385.

Thus, $\frac{m_1}{l_1}$, $\frac{m_2}{l_2}$ are given as the roots of the equation

$$bm^2 - 2hlm + al^2 = 0.$$

Similarly, if we find the corresponding equations for the other two pairs of tangents, we see that these six tangents touch the conic—

$$al^2 + bm^2 + cn^2 - 2fmn - 2gnl - 2hlm = 0.$$

For, if in this equation we put $n=0$, we shall obtain the equations of the tangents which can be drawn from tbe vertex C and these are found to be the same as those satisfied by the parameters of the nodal tangents at C.

295. *From each node of a trinodal quartic, two tangents can be drawn to the curve, and these six tangents touch a conic.*

Consider the tangents which can be drawn from the points A, B, C to a given conic. Then, if we take the inverse, it can be proved that the inverse lines also touch a conic. This is because the parameters of the six tangents satisfy a relation which remains unchanged when the parameters are replaced by their reciprocals. Now, transforming the conic into a trinodal quartic, the tangents drawn from A to the conic are transformed into the tangents from the node A to the quartic. Therefore these tangents all touch a conic.

296. From Plücker's formulae, it is seen that a trinodal quartic has four bitangents and six points of inflexion. Now, to the bitangents of the quartic correspond conics through A, B, C, having double contact with the conic ; and to the stationary tangents of the quartic

correspond conics through A, B, C, osculating the conic. Thus we obtain the theorem :—

Through three given points there can be drawn four conics which have double contact with a given conic, and six conics which osculate a given conic.

It is to be remarked here that the four conics having double contact determine a complete quadrilateral, of which the fundamental triangle is the diagonal triangle. From this fact they can easily be constructed geometrically.

297. *The eight points of contact of the four double tangents of a trinodal quartic lie on a conic.*

Note: This conic is called the bitangential conic.

Let $t_1=0$, $t_2=0$, $t_3=0$, $t_4=0$ be the equations of the four double tangents. Then the equation of the quartic can be written as

$$t_1^{\frac{1}{2}}+t_2^{\frac{1}{2}}+t_3^{\frac{1}{2}}+t_4^{\frac{1}{2}}=0 \qquad \dots \quad (1)$$

which may be put into the rationalised form—

$$(t_1^2+t_2^2+t_3^2+t_4^2-2t_1t_2-2t_2t_3-2t_3t_4-2t_1t_4-2t_1t_3 \\ -2t_2t_4)^2=64t_1t_2t_3t_4.$$

From this form it is easily seen that t_1, t_2, t_3, t_4 are bitangents to the quartic, whose points of contact lie on a conic.

The curve (1) has also three double points. The equation can be written as

$$[(t_1-t_3)+(t_2-t_4)]^4-8[(t_1-t_3)+(t_2-t_4)]^2(t_1t_2+t_3t_4) \\ +16[(t_1-t_3)\,t_2+(t_2-t_4)\,t_3]^2=0.$$

Each term in this form contains either $(t_1-t_3)^2$ or $(t_1-t_3)\,(t_2-t_4)$ or $(t_2-t_4)^2$.

Hence, the point $t_1-t_3=0$, $t_2-t_4=0$ is a double point.

Similarly, the points (t_1-t_2, t_3-t_4) and (t_1-t_4, t_2-t_3) are also double points.

298. The properties of stationary points and tangents cannot so easily be deduced from known properties of conics. There exists a complete theory of the 28 bitangents of a quartic, but very little is still known of the 24 stationary points. Some of the properties of trinodal quartics have been given by A. Brill,* the most important of these are the following:—

(1) *The six points of inflexion of a trinodal quartic lie on a conic.*†

(2) *The six points of contact of the tangents drawn from the nodes lie on a second conic.*

(3) *The six points in which the nodal tangents intersect the quartic lie on a third conic.*

299. *The six stationary tangents of a trinodal quartic touch a conic.*‡

Let the equation of the quartic be—

$$ay^2z^2+bz^2x^2+cx^2y^2+2fx^2yz+2gy^2zx+2hz^2xy=0 \quad \dots (1)$$

and that of a tangent be— $$\lambda x+\mu y+\nu z=0 \quad \dots (2)$$

Now, writing $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ for x, y, z in these two equations, we obtain—

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0, \quad \dots (3)$$

$$\lambda yz+\mu zx+\nu xy=0. \quad \dots (4)$$

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* A. Brill—Über rationale Kurven vierter Ordnung—Math. Ann. Bd. XII, pp. 89-90.

† This theorem was originally given by J. Grassmann in his Dissertation—Zur Theorie der Wendepunkte etc. (Berlin, 1875.)

‡ This theorem is due to N. M. Ferrers, Quart. Journal Vol. XVIII p. 73.

Therefore, if (2) be a stationary tangent to (1), *i.e.* if (2) meets (1) in three consecutive points, then the two conics (3) and (4) must osculate. This also follows from the fact that to a stationary tangent corresponds a conic through the points A, B, C, osculating the conic.

But the conditions* that the two conics (3) and (4) may have a contact of the second order are given by

$$\frac{3\Delta}{\Theta}=\frac{\Theta}{\Theta'}=\frac{\Theta'}{3\Delta'} \qquad \ldots \qquad \ldots \ (5)$$

where $\Delta=abc+2fgh-af^2-bg^2-ch^2$

$$\Delta'=2\lambda\mu\nu$$

$$\Theta=2(gh-af)\lambda+2(hf-fg)\mu+2(fg-ch)\nu,$$

$$\Theta'=-a\lambda^2-b\mu^2-c\nu^2+2f\mu\nu+2g\nu\lambda+2h\lambda\mu.$$

The equations (5) are equivalent to

$$3\Delta\Theta'=\Theta^2 \text{ and } 3\Delta'\Theta=\Theta'^2.$$

The first of these equations is an equation of the second degree in λ, μ, ν and consequently represents tangentially a conic touched by the line $\lambda x+\mu y+\nu z=0$. Therefore the six stationary tangents touch the conic $3\Delta\Theta'=\Theta^2$. The second equation represents a curve of the fourth class, also touched by the six stationary tangents.

300. The theory of trinodal quartics extends to the case when any or all of the double points are cusps. If all the double points are cusps, the equation of a tri-cuspidal quartic having A, B, C for cusps can be written as

$$y^2z^2+z^2x^2+x^2y^2-2x^2yz-2y^2zx-2z^2xy=0,$$

which can again be written as—

$$\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}+\frac{1}{\sqrt{z}}=0.$$

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* Salmon—Conic Sections § 372.

The tangents at the cusps are $x=y$, $y=z$, $z=x$, which evidently meet in a point.

Writing $\frac{1}{X}$, $\frac{1}{Y}$, $\frac{1}{Z}$ for x, y, z, the equation becomes

$$X^2+Y^2+Z^2-2YZ-2ZX-2XY=0,$$

which can be written as—

$$(X+Y-Z)^2=4XY.$$

Let $$X+Y-Z=2\mu X$$

$$\therefore 4\mu^2X^2=4XY, \text{ or, } \mu^2X=Y.$$

Also, $$2\mu X(X+Y-Z)=4XY$$

or, $$\mu(X+Y-Z)=2Y$$

$$\therefore (1-\mu)^2X=Z.$$

$$\therefore X:Y:Z=1:\mu^2:(1-\mu)^2.$$

$$\therefore x:y:z=1:\frac{1}{\mu^2}:\frac{1}{(1-\mu)^2}$$

$$=\mu^2(1-\mu)^2:(1-\mu)^2:\mu^2.$$

Thus the co-ordinates of any point on a tri-cuspidal quartic can be expressed in terms of a parameter.

Hence, the properties of trinodal quartics also hold for tricuspidal quartics.

CHAPTER XVIII.

Bicircular Quartics.

301. Definition: A binodal (or trinodal) quartic which has two nodes at the circular points at infinity is called a *bicircular quartic.* This class of quartics belongs to species IV, VII or VIII.

There is another class of quartics which has a pair of cusps at the circular points, and belongs to the species VI, IX or X. They are called *Cartesians.* The Oval of Descartes was the first curve studied under this class.

302. *To find the general Cartesian equation of a bicircular quartic.*

The general equation of a quartic curve is

$$u_0+u_1+u_2+u_3+u_4=0.$$

We have seen that $u_4=0$ represents four right lines drawn through the origin, which are parallel to the asymptotes of the curve.

Now, the circular lines through the origin are $y\pm ix=0$. If the curve passes through the circular points I and J, $y\pm ix=0$ are parallel to the asymptotes whose points of contact are at I or J. If further I and J are nodes on the curve, the nodal tangents at each of I and J reduce to two coincident tangents, which are consequently two coincident asymptotes of the curve. Hence, the four asymptotes reduce to a pair of coincident asymptotes, parallel to $y\pm ix=0$.

Thus, $$u_4 \equiv (y\pm ix)^2 = (x^2+y^2)^2.$$

Again, each of the lines $y\pm ix=0$ must meet the curve in two coincident points at I or J.

Let $u_3 \equiv ax^3+3bx^2y+3cxy^2+dy^3$. Substituting $y=\pm ix$ in the equation of the curve, we find

$$0.\, x^4+(a\pm 3ib-3c\mp id)x^3+const.\, x^2+const.\, x+u_0=0.$$

Now, two roots of this equation should be infinite *i.e.* the co-efficients of the two highest powers of x should vanish.

Therefore, $$a\pm 3ib-3c\mp id=0.$$

Hence $3b=d$ and $3c=a$. $\therefore\ u_3\equiv(x^2+y^2)\,(ax+dy)$.

Thus the equatiou of a quartic having two nodes at I and J is

$$A(x^2+y^2)^2+(x^2+y^2)(ax+dy)+u_2+u_1+u_0=0\,;$$

or, in the symbollic form—

$$u_0r^4+u_1r^2+v_2+v_1+v_0=0. \qquad \dots \quad (1)$$

This equation contains eight independent constants, as it should be; for the general equation of a quartic contains 14 disposable constants; the condition for a double point is equivalent to three conditions and the fact that I and J are nodes on the curve amounts to six conditions.

303. The above equation can be written in the form

$$r^4+2u_1r^2+v_2+v_1+v_0=0.$$

Adding and subtracting u_1^2 to the equation, we obtain—

$$(r^4+2u_1r^2+u_1^2)=u_1^2-v_2-v_1-v_0,$$

or,

$$(r^2+u_1)^2=u_1^2-v_2-v_1-v_0.$$

Now, the left-hand side represents a circle and the right-hand side is the equation of a conic. Hence the

equation of a bicircular quartic can be written in the form $C^2 = S$, where C is a circle and S is any conic, or in the form $C^2 = SI$, where $I = 0$ is the line at infinity. This last equation shows that the conic S and the line at infinity I have a contact of the first order with the quartic, at the two points where it is cut by the circle C. This circle has also a contact of the first order with the quartic, at the two points where it meets the line at infinity, *i.e.* at the two circular points at infinity. The conic S touches the quartic at the four points where it cuts the circle C. It is to be noticed that the contact of the circle and the line at infinity with the curve is due to their passing through I and J, which are two nodes on the curve.

304. The converse theorem is also true that, if the equation of a quartic can be brought to the form $C^2 = S$, where C is a circle and S any conic, the curve is a bicircular quartic.

Again, $$(C+k)^2 = S + 2kC + k^2$$

$$\therefore \; C'^2 = S', \text{ where } C' = C + k \text{ i.e. a circle}$$

and $$S' = S + 2kC + k^2 \text{ i.e. a conic.}$$

Now, since there is only one constant k involved in this reduction, the equation can be reduced to the above form in a singly infinite number of ways. The circles $C + k$ evidently have the same centre for all values of k. ∴ These circles are concentric.

305. Quartics with two nodes, in the case where these are at the two circular points at infinity, have been exhaustively studied by Dr. Casey* under the name of bicircular quartics. Most of the results given by him will form the subject matter of the present discourse.

* Casey's Memoir—Transactions of the Royal Irish Academy, Vol. XXIV, p. 457.

We have seen that the equation of a quartic can be reduced to the form $UW = V^2$, where U, V, W are three conics. Now these three conics cannot have a common point when the quartic is non-singular; for, in that case the point must be a double point on the quartic. But in the case of binodal quartics, U, V, W may be taken as three conics passing each through the two nodes. If the two nodes are at the circular points at infinity, U, V, W are all circles. Hence we may regard a bicircular quartic as the envelope of the variable circle $\lambda^2 U + 2\lambda V + W = 0$, where U, V, W are circles.

Again, we have shown that $\lambda^2 U + 2\lambda V + W = 0$ touches the quartic at the four points $\lambda U + V = 0$, $\lambda V + W = 0$. Now, when U, V, W are all circles, they pass through the two circular points at infinity. Hence, besides the two points I and J at infinity, the two circles $\lambda U + V$ and $\lambda V + W$ intersect in two other finite points, which are the points of contact of the enveloping circle $\lambda^2 U + 2\lambda V + W = 0$ with the bicircular quartic. Therefore, each of the variable circles $\lambda^2 U + 2\lambda V + W = 0$ has double contact with the bicircular quartic at the two points given by

$$\lambda U + V = 0 \text{ and } \lambda V + W = 0.$$

Hence the chord of contact of the variable circle $\lambda^2 U + 2\lambda V + W = 0$ with the quartic is the radical axis of the two circles $\lambda U + V = 0$, $\lambda V + W = 0$. Now, if L, M, N be the linear parts in the equations of U, V, W respectively, the radical axis is $\lambda(L - M) + (M - N) = 0$, which evidently passes through the fixed point $L = M = N$, which is the radical centre of the three circles U, V, W. Thus the enveloping circle has double contact with the bicircular quartic, the chord of contact always passing through a fixed point.

306. It is proved in Treatises on Conics* that the Jacobian of three circles consists of the line at infinity and another circle which cuts them all orthogonally. Also the Jacobian of three conics (circles in particular), whose equations are of the form $l\mathrm{U}+m\mathrm{V}+n\mathrm{W}=0$, is the same as that of U, V, W. Hence, when U, V, W are all circles, the circles included in the form $l\mathrm{U}+m\mathrm{V}+n\mathrm{W}=0$, (and in particular the form $\lambda^2\mathrm{U}+2\lambda\mathrm{V}+\mathrm{W}=0$) have a common Jacobian, and consequently a common orthogonal circle. Thus the enveloping circles always cut a fixed circle orthogonally.

Again, if (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) are the co-ordinates of the centres of the three circles U, V, W, then the co-ordinates of the centre of the circle $\lambda^2\mathrm{U}+2\lambda\mathrm{V}+\mathrm{W}=0$ are $\lambda^2x_1+2\lambda x_2+x_3$, $\lambda^2y_1+2\lambda y_2+y_3z$, $\lambda^2{}_1+2\lambda z_2+z_3$. If we denote these co-ordinates by x, y, z,

$$\left.\begin{aligned} x&=\lambda^2x_1+2\lambda x_2+x_3\\ y&=\lambda^2y_1+2\lambda y_2+y_3\\ z&=\lambda^2z_1+2\lambda z_2+z_3.\end{aligned}\right\}$$

Eliminating λ^2 and λ between these equations, we obtain the locus of the centre x, y, z as a conic. Thus we obtain the theorem :

A bicircular quartic is the envelope of a variable circle whose centre moves along a fixed conic and which cuts a fixed circle orthogonally.

307. This may be shown directly as follows :—

Let $$x^2+y^2+2fx+2gy+c=0 \quad \ldots \quad \ldots \quad (1)$$

be the fixed circle

and $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \ldots \quad \ldots \quad (2)$$

be the fixed conic.

* Salmon—Conics § 388, *Ex.* 3.

Any point on the conic is $(a\cos\theta,\ b\sin\theta)$.

The equation of a variable circle whose centre is $(a\cos\theta,\ b\sin\theta)$ may be taken as—

$$(x-a\cos\theta)^2+(y-b\sin\theta)^2=r^2, \text{ where } r \text{ varies.} \quad \ldots \quad (3)$$

Now, the condition that the circle (3) cuts (1) orthogonally gives us

$$(a\cos\theta+f)^2+(b\sin\theta+g)^2=(f^2+g^2-c)+r^2$$

or

$$r^2=c+2af\cos\theta+2bg\sin\theta+a^2\cos^2\theta+b^2\sin^2\theta \quad \ldots \quad (4)$$

Eliminating r^2 between (3) and (4), we obtain the equation of a variable circle which cuts (1) orthogonally and whose centre moves along (2).

The equation of the variable circle thus becomes

$$x^2+y^2-2ax\cos\theta-2by\sin\theta=c+2af\cos\theta+2b\sin\theta,$$

or,

$$2a(x+f)\cos\theta+2b(y+g)\sin\theta=x^2+y^2-c \quad \ldots \quad (5)$$

i.e. of the form $L\cos\theta+M\sin\theta=N$, where L, M, N are functions of co-ordinates.

In order to determine the envelope of (5) as θ varies, we differentiate it with respect to θ. Thus

$$-L\sin\theta+M\cos\theta=0. \quad \ldots \quad (6)$$

Squaring and adding, we obtain $L^2+M^2=N^2$ as the envelope.

Thus the envelope of the variable circle (5) is—

$$4a^2(x+f)^2+4b^2(y+g)^2=(x^2+y^2-c)^2 \quad \ldots \quad (A)$$

which is of the form $C^2=S$, where C is a circle and S any conic, and it therefore represents a bicircular quartic.

308. The equation of the above quartic, referred to $(-f, -g)$ the centre of the fixed circle, is—

$$4a^2x^2+4b^2y^2=\{x^2+y^2-2fx-2gy+\delta^2\}^2,$$

where δ is the radius of the fixed circle.

i.e. $$4(a^2x^2+b^2y^2)=(r^2-u_1+\delta^2)^2 \quad \dots \quad (1)$$

The fixed conic is called the *focal conic*; for, as will be shown later on, it passes through four foci of the quartic.

If the quartic be inverted from the centre $(-f, -g)$ with respect to the fixed circle, it is inverted into itself. For, the inverse of (1) is

$$4\frac{\delta^4}{r^4}(a^2x^2+b^2y^2)=\left\{\frac{\delta^4}{r^4}r^2-\frac{\delta^2}{r^2}u_1+\delta^2\right\}^2$$

or, $$4(a^2x^2+b^2y^2)=(\delta^2-u_1+r^2)^2,$$

which is the same equation as (1).

It is on this account that the centre $(-f, -g)$ of the fixed circle is called the *centre of inversion*, and the fixed circle of radius δ is called the *circle of inversion*. The variable circle is called the *generating circle*.

309. *There are four circles of inversion and fonr corresponding focal conics associated with any bicircular quartic.*

We have seen that the quation of a bicircular quartic is $C^2=S$, where C is a circle and S any conic. Also the circles C are concentric. Let the centre of this system be taken as origin and the axes be parallel to the principal axes of the conic S. Then the equation of **S** is of the form $S \equiv a'x^2+b'y^2+2f'x+2g'y+c'=0$. Let C be a circle of the system of zero radius, so that $C \equiv x^2+y^2$.

Then the equation of the quartic can be written as

$$a'x^2+b'y^2+2f'x+2g'y+c'=(x^2+y^2)^2 \quad \dots \quad (1)$$

or $$(a'-2\lambda)x^2+(b'-2\lambda)y^2+2f'x+2g'y+c'+\lambda^2$$
$$=(x^2+y^2-\lambda)^2 \quad \ldots \ (2)$$

Now, this equation can be written in the form

$$(a'-2\lambda)\left\{x^2+\frac{2f'}{a'-2\lambda}x+\frac{f'^2}{(a'-2\lambda)^2}\right\}$$
$$+(b'-2\lambda)\left\{y^2+\frac{2g'y}{b'-2\lambda}+\frac{g'^2}{(b'-2\lambda)^2}\right\}$$
$$+\left\{c'+\lambda^2-\frac{f'^2}{a'-2\lambda}-\frac{g'^2}{b'-2\lambda}\right\}=(x^2+y^2-\lambda)^2,$$

or, $$(a'-2\lambda)\left\{x+\frac{f'}{a'-2\lambda}\right\}^2+(b'-2\lambda)\left\{y+\frac{g'}{b'-2\lambda}\right\}^2$$
$$+\left\{c'+\lambda^2-\frac{f'^2}{a'-2\lambda}-\frac{g'^2}{b'-2\lambda}\right\}=(x^2+y^2-\lambda)^2 \quad \ldots \ (3)$$

Thus, if this is to be of the same form as (A) of § 307, we must have $c'+\lambda^2-\dfrac{f'^2}{a'-2\lambda}-\dfrac{g'^2}{b'-2\lambda}=0$, which gives four values of λ. Hence there are four different ways of reducing the equation of a bicircular quartic to the form (A).

Corresponding to each value of λ, we obtain a circle of inversion and a corresponding focal conic. By comparing the equation (3) with (A) of § 307, we obtain

$$4a^2=a'-2\lambda\,;\ 4b^2=b'-2\lambda\,;\ f=\frac{f'}{a'-2\lambda},\ g=\frac{g'}{b'-2\lambda},\ c=\lambda.$$

Therefore the equation of the focal conic is

$$\frac{4x^2}{a'-2\lambda}+\frac{4y^2}{b'-2\lambda}=1 \quad \ldots \quad \ldots \quad (4)$$

which represents a system of confocal conics.

The equation of the circle of inversion is

$$x^2+y^2+\frac{2f'x}{a'-2\lambda}+\frac{2g'y}{b'-2\lambda}+\lambda=0 \quad \ldots \quad (5)$$

which represents a circle whose centre is the point

$$\left(-\frac{f'}{a'-2\lambda},\ -\frac{g'}{b'-2\lambda}\right)$$

Hence the four centres of inversion are obtained as the intersection of two rectangular hyperbolas which are obtained by eliminating λ between

$$x=\frac{-f'}{a'-2\lambda},\ y=\frac{-g'}{b'-2\lambda} \text{ and } c'+\lambda^2-\frac{f'^2}{a'-2\lambda}-\frac{g'^2}{b'-2\lambda}=0,$$

i.e., the four centres of inversion are the intersections of the two rectangular hyperbolas $(a'-b')\ xy+fy-g'x=0$ and

$$4f'g'(x^2-y^2)+4(g'^2-f'^2)xy+g'(4c'-a'^2)x$$
$$+f'(b'^2-4c')y-(a'-b')f'g'=0.$$

Therefore, by a known theorem in conics, it follows that the four centres of inversion are such that each is the orthocentre of the triangle formed by the other three.

Hence a bicircular quartic may be regarded as *the envelope of a variable circle which cuts any of the four fixed circles orthogonally and whose centre moves along any of the four corresponding confocal conics (focal conics). The centres of these four fixed circles are such that each is the orthocentre of the triangle formed by the other three.*

310. Let P, Q, R, S be the four points in which a circle of inversion intersects the corresponding focal conic. Let PR, QS ; PQ, RS ; and PS, QR respectively intersect in the three points A, B, C. It is proved in conics that the triangle ABC is self-conjugate with respect to all

conics through PQRS. Hence it is self-conjugate with respect to the circle of inversion and consequently the centre of the circle is the orthocentre D of the triangle ABC, and the four points A, B, C, D are such that each is the orthocentre of the triangle formed by the other three. (Fig. 36).

Now, with centres A, B, C describe three circles U, V, W such that the orthocentre D is the radical centre of the three circles. Then the circle of inversion intersects each of U, V, W orthogonally, and consequently $lU + mV + nW = 0$, or in particular, $\lambda^2 U + 2\lambda V + W = 0$ also. Hence the bicircular quartic may be generated as the envelope of the variable circle $lU + mV + nW = 0$, or in particular, of $\lambda^2 U + 2\lambda V + W = 0$, which cuts the circle of inversion I, as well as each of the circles U, V, W orthogonally and whose centre moves along the focal conic.

The equation of the bicircular quartic can, in this case, be written as $\lambda U \ + \mu V^2 + \nu W^2 = 0$.

These four circles I, U, V, W cut mutually at right angles and the four centres A, B, C, D are such that the four triangles formed by each group of three have the same nine-points circle, and the radical axis of any two passes through the centres of the remaining two.

It follows therefore that the bicircular quartic can also be generated as the envelope of a variable circle which cuts each of the four circles I, U, V, W orthogonally and whose centre moves along a corresponding focal conic.

311. We shall now show that the three points A, B, C are also centres of inversion of the bicircular quartic.

Since each of the circles U, V, W intersects I orthogonally, it follows that the radii of U, V, W are the tangents drawn to I from their centres A, B, C. Let r_1, r_2, r_3 be these radii respectively.

Let A be the origin and AB the axis of x, and let $BC=a$, $CA=b$, $AB=c$. Then the co-ordinates of B and C are $(c, 0)$, $(b \cos A, b \sin A)$

$\therefore$ The equations of U, V, W are respectively—

$$U=x^2+y^2-r_1^2=0,$$

$$V=(x-c)^2+y^2-r_2^2=x^2+y^2-2cx+c^2-r_2^2=0,$$

$$W=(x-b \cos A)^2+(y-b \sin A)^2-r_3^2=0$$

$$=x^2+y^2-2b(x \cos A+y \sin A)+(b^2-r_3^2)=0.$$

But $a^2=r_2^2+r_3^2$; $b^2=r_3^2+r_1^2$; $c^2=r_1^2+r_2^2$.

$$\therefore \quad U=x^2+y^2-r_1^2=0, \; V=x^2+y^2-2cx+r_1^2=0,$$

$$W=x^2+y^2-2b(x \cos A+y \sin A)+r_1^2=0.$$

Now, the equation of the quartic is

$$\lambda U^2+\mu V^2+\nu W^2=0.$$

Let us invert this with respect to the circle U.

Now, if the inverses of U, V, W are respectively U′, V′, W′, then,

$$r^2 U'=-r_1^2 U; \quad r^2 V'=r_1^2 V; \quad r^2 W'=r_1^2 W.$$

Consequently, the quartic $\lambda U^2+\mu V^2+\nu W^2=0$ is its own inverse, which shows that A is a centre of inversion.

312. We shall now prove that *the four points of intersection of the circle of inversion with the corresponding focal conic are the foci of the bicircular quartic.*

Consider the variable circle whose centre is the point P and which cuts the circle of inversion orthogonally.

Let r be the radius of this circle and r' that of the fixed circle. Since the two circles are orthogonal, we must have $r'^2 = r'^2 + r^2$. $\therefore r = 0$, or the variable circle is of zero radius, *i.e.* a point circle. But each generating circle has double contact with the quartic (§ 305). Hence, at P we have a point circle which has double contact with the curve, *i.e.* the point P is a focus. Similarly, Q, R, S are also foci of the bicircular quartic.

Now, we have seen that there are four circles of inversion and four corresponding focal conics associated with any bicircular quartic. Hence we obtain Dr. Hart's Theorem, namely, *there are sixteen foci of a bicircular quartic, which lie on four circles, four on each circle.*

313. This theorem of Dr. Hart's can be deduced from a more general theorem on binodal quartics, namely,—*The anharmonic ratios of the two pencils of four tangents, which can be drawn to the curve from the two nodes, are equal.**

The equation of a quartic having two nodes at the vertices B and C of the triangle of reference is—

$$y^2z^2 + 2xyz(my + nz) + x^2(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) = 0.$$

The pairs of tangents at B and C are respectively given by—

$$z^2 + 2mzx + bx^2 = 0 \text{ and } y^2 + 2nxy + cx^2 = 0.$$

Now choose y and z to be the harmonic conjugates of x with respect to the nodal tangents at C and B respectively. This requires that $m = 0$ and $n = 0$ and the nodal tangents are

$$z^2 + bx^2 = 0 \text{ and } y^2 + cx^2 = 0.$$

* *Cf.* Cayley—“Memoir on Polyzomal curves”—Edinburgh Transactions. Coll. Papers, Vol. VI, p. 529 (1868).

The equation o the quartic becomes

$$f=y^2z^2+x^2(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)=0. \quad (1)$$

The equation of the four tangents drawn from the point B is obtained by the method of § 51. Or we may find the equation of the tangents from B by first writing down the equation of the first polar of B and then eliminating y between this and the equation of the curve. Now, the first polar of B is $\frac{df}{dy}=0$. Therefore, the result of elimination of y between $\frac{df}{dy}=0$ and $f=0$ is the same as the condition for a double root of the equation $f=0$, regarded as an equation in y. Thus, the equation of the tangents is

$$(z^2+bx^2)(ax^2+2gzx+cz^2)=x^2(fz+hx)^2$$

or
$$x^4(ab-h^2)+(2bg-2fh)x^3z+x^2z^2(bc+a-f^2)$$
$$+xz^3.2g+cz^4=0 \quad \ldots \quad (2)$$

Now, the six anharmonic ratios of this pencil are given by

$$\frac{I^3*}{27J^2}=\frac{(\sigma^2-\sigma+1)^3}{(\sigma+1)^2\ (\sigma-2)^2\ (\sigma-\frac{1}{2})^2} \quad \ldots \quad (3)$$

where I and J are the invariants of (2).

Now, $I=c(ab-h^2)-4.\frac{2(bg-fh)}{4}.\frac{2g}{4}+3.\frac{(ca+b-f^2)^2}{36}$

$$=abc-ch^2-g\,(bg-fh)+\tfrac{1}{12}\,(a+bc-f^2)^2$$

$$=abc-ch^2-bg^2+fgh+\tfrac{1}{12}\,(a+bc-f^2)^2.$$

and $J=(ab-h^2)\frac{(a+bc-f^2)}{6}.c+2.\frac{2(bg-fh)}{4}.\frac{(a+bc-f^2)}{6}.\frac{2g}{4}$

$$-(ab-h^2)\frac{4g^2}{16}-c.\frac{4(bg-fh)^2}{16}-\frac{(a+bc-f^2)^3}{216}$$

* Burnside and Panton—Theory of Equations, Vol. I, Ex. 16, pp. 148-150.

$$\therefore 6\mathrm{J}=(abc-ch^2)(a+bc-f^2)+\tfrac{1}{2}(bg^2-fgh)(a+bc-f^2)$$
$$-\tfrac{3}{2}g^2(ab-h^2)-\tfrac{3}{2}c(bg-fh)^2-\tfrac{1}{36}(a+bc-f^2)^3.$$
$$=(abc-ch^2-bg^2-\tfrac{1}{2}fgh)\,(a+bc-f^2)-\tfrac{3}{2}f^2(ch^2+bg^2)$$
$$+3bcfgh+\tfrac{3}{2}g^2h^2-\tfrac{1}{36}\,(a+bc-f^2)^3.$$

Now we see that I and J are symmetrical between b and c, as well as g and h; and consequently the equation (3) remains unchanged when we interchange b and c, as also g and h. But, by this interchange we obtain only the equation of the four tangents drawn from C. Hence the anharmonic ratios of the two pencils are equal.

314. Since the two pencils are homographic, a conic can be described through the points of intersection of the corresponding rays of the two pencils, which also passes through the nodes. Again, since there are four orders in which the legs of the second pencil can be taken without altering the anharmonic ratio, it follows that the sixteen points of intersection of the first pencil with the second lie on four conics, each passing through the two nodes. But when the quartic becomes bicircular, the nodes are at the circular points I and J, and the conics become circles; also the points of intersection of the tangents become the foci of the curve. Hence the theorem becomes that *the sixteen foci of a bicircular quartic lie on four circles, four on each circle,* as is otherwise shown in § 312.

315. We have seen that the generating circle $\lambda^2\mathrm{U}+2\lambda\mathrm{V}+\mathrm{W}=0$ has double contact with the bicircular quartic, the chord of contact passing through the radical centre of U, V, W. But the radical centre of U, V, W is a centre of inversion. Hence the chord of contact passes through a centre of inversion.

Again, there are four centres of inversion and four systems of generating circles. Hence we obtain the general

theorem:—*If a circle has double contact with a bicircular quartic, its chord of contact must pass through one of four fixed points.*

An analytical proof of this theorem may be given as follows:—The equation of the generating circle is (§ 307)

$$2a(x+f)\cos\theta+2b(y+g)\sin\theta=x^2+y^2-c \quad \ldots \quad (1)$$

This touches the quartic at the two points where it is met by the line $2b(y+g)\cos\theta-2a(x+f)\cos\theta=0 \quad \ldots \quad (2)$

which for all values of θ passes through the fixed point $x=-f$, $y=-g$, which is the centre of inversion.

316. Cyclic Points. If the line (2) is a tangent to (1), it will also be a tangent to the bicircular quartic at a point where the circle (1) has a third order contact with the curve, *i.e.* the point of contact is a *cyclic* point on the curve. Now, the line (2) touches (1), when the perpendicular from the centre on the line = radius of the circle.

This gives certain definite values of θ, and therefore the curve has a finite number of cyclic points, which are the points of contact of the tangents drawn from a centre of inversion. Hence, *the points of contact of tangents drawn from a centre of inversion to a bicircular quartic are the cyclic points on the curve.*

Again, since the generating circle (1) intersects the circle of inversion orthogonally, whose centre lies on the line (2), it follows that, when the line (2) touches the circle (1), the point of contact lies on the circle of inversion and this point is a cyclic point on the quartic. Hence the four finite points in which the circle of inversion cuts the bicircular quartic are cyclic points on the latter.

Since there are four circles of inversion, these circles intersect the quartic in sixteen finite points, which are cyclic points on the curve. The same thing also follows from the fact that, from each centre of inversion four ordinary tangents can be drawn to the curve, whose points of contact are cyclic points. [We have seen that these points also lie on the circles of inversion. Therefore the lengths of the four tangents drawn from a centre of inversion are equal.] Thus we have the theorem : *each bicircular quartic has sixteen cyclic points which lie on four circles, four on each circle.*

317. *The inverse of a bicircular quartic is another bicircular quartic ; but when the origin of inversion lies on the curve, the inverse is a circular cubic.*

The general equation of a bicircular quartic can be written as $u_0 r^4 + u_1 r^2 + v_2 + v_1 + v_0 = 0$. The inverse of this with respect to the origin is $v_0 r^4 + k^2 v_1 r^2 + k^4 v_2 + k^6 u_1 + k^8 u_0 = 0$, which is also a bicircular quartic. If the origin lies on the curve, $v_0 = 0$, and the inverse becomes $v_1 r^2 + k^2 v_2 + k^4 u_1 + k^6 u_0 = 0$, which is a circular cubic.

318. *The foci of the focal conic are the double foci of the bicircular quartic.*

Let $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ be the focal conic and

$x^2 + y^2 + 2fx + 2gy + c = 0$ be the circle of inversion.

Then the equation of the bicircular quartic is—

$$4a^2(x+f)^2 + 4b^2(y+g)^2 = (x^2+y^2-c)^2 \qquad \dots \quad (1)$$

Now, the double foci of a curve are the real points of intersection of the nodal tangents at the two circular points I and J.

Any line through I is $y=ix+k$ or $y+ix=2ix+k$... (2)

In order to determine the points where (2) intersects (1), we have—

$$4a^2(x+f)^2+4b^2(ix+k+g)^2=\{k(2ix+k)-c\}^2. \quad \ldots \quad (3)$$

If the line is a nodal tangent at I, three roots of equation (3) should be infinite and consequently the co-efficients of x^4, x^3 and x^2 must vanish. It is evident that for all values of k, the co-efficients of x^4 and x^3 are zero. The co-efficient of x^2 will be zero,

if, $\quad 4a^2-4b^2+4k^2=0 \qquad$ *i.e.*, if, $a^2-b^2=-k^2$.

$$\therefore \quad k=\pm i\sqrt{a^2-b^2}.$$

Therefore, the equation of the nodal tangents at I are

$$y=i(x+\sqrt{a^2-b^2}) \text{ and } y=i(x-\sqrt{a^2-b^2}) \ \ldots \quad (4)$$

Similarly, the nodal tangents at J are—

$$y=-i(x\pm\sqrt{a^2-b^2}) \quad \ldots \quad \ldots \quad (5)$$

$\therefore$ The double foci are the real points of intersection of (4) and (5), *i.e.* the points $y=0, x=\pm\sqrt{a^2-b^2}=\pm ae$, the common foci of the system of focal conics, are the double foci of the bicircular quartic.

319. We have seen that the inverse of a bicircular quartic with respect to a circle of inversion is the curve itself, and the foci of a curve are also the foci of its inverse. Hence, if we invert a bicircular quartic with respect to a circle of inversion, the sixteen foci will be inverted into the sixteen foci of the inverse curve, but each iudividual focus is not inverted into itself. Let us

investigate the relation between these foci and their inverse points.

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the foci on the circle with centre A. Similarly, let $(\beta_1, \beta_2, \beta_3, \beta_4)$, $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ and $(\delta_1, \delta_2, \delta_3, \delta_4)$ be the groups of foci on the circles with centres B, C and D respectively.

Invert the curve with respect to the circle A. Then the foci $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are inverted into themselves. The inverse of β_1, a focus on B, will be a focus of the inverse curve, *i.e.* a focus of the same curve. Therefore, the inverse of β_1 must be one of the remaining 11 foci. Thus the line $A\beta_1$ must pass through one of these remaining foci. Three different cases are to be considered :—

The inverse of β_1 may be—

(1) one of the three points $\beta_2, \beta_3, \beta_4$,

(2) one of the points of the γ group ;

(3) one of the points of the δ group.

(1) When the inverse of β_1 is one of the points $\beta_2, \beta_3, \beta_4$, the circle B is inverted into itself, and consequently the γ group is inverted into the δ group, or they are the inverses of themselves.

(2) When β_1 is inverted into one of the foci of the γ group, then the circle B is inverted into the circle C, and consequently the points A,B,C are collinear. In this case, the δ group cannot be inverted into any of the three groups α, β, or γ. For, the α group is its own inverse and β group inverts into the γ group. Consequently the δ group is inverted into itself.

(3) In this case β group is inverted into the δ group, while the γ group is inverted into itself.

Now, since the bicircular quartic has four circles of inversion, any property which holds for any two must also hold for the remaining two. The relation between the four circles must be symmetrical. Hence, the only symmetrical relation which may exist between them is that each of the groups is inverted into itself. This will be true also for the other three centres of inversion. Hence, the four circles of inversion are inverted into themselves, when inverted with respect to any one of them. (§ 311.)

320. When A is the centre of inversion, a_1, a_2, a_3, a_4 are inverted each into itself. But δ_1 cannot be inverted into itself, and since the circle D is inverted into itself, the inverse of δ_1 will be one of the three foci δ_2, δ_3, δ_4 (say δ_2). Then A, δ_1, δ_2 are in one right line. Consequently A, δ_3, δ_4 are in one right line.

Thus, A is a diagonal point of the quadrangle $\delta_1\delta_2\delta_3\delta_4$. Similarly, if B is the centre of inversion, δ_1, δ_2, δ_3, δ_4 are inverted into themselves and consequently B is collinear with $\delta_1\delta_4$ and $\delta_2\delta_3$. Hence B is a diagonal point of the same quadrangle. Similarly, the centre C is the third diagonal point of $\delta_1\delta_2\delta_3\delta_4$. Also, D is the centre of the circle $\delta_1\delta_2\delta_3\delta_4$. (Fig. 37).

Hence ABC is self-conjugate with respect to the circle whose centre is D. Therefore D is the orthocentre of the triangle ABC.

Similarly, it can be shown that any of the four points A,B,C,D, will be the orthocentre of the triangle formed by the other three, as was otherwise shown in § 310.

321. From what has been said above it follows that the circles of inversion cut each other orthogonally. For, consider the circles A and D. δ_1 and δ_2 are inverse points with respect to the circle A. $\therefore$ $A\delta_1.A\delta_2=r^2$, where r is the radius of the circle A.

But $A\delta_1 . A\delta_2$ = square of the length of the tangent drawn from A to the circle D.

$\therefore$ $A\delta_1 . A\delta_2 = r^2$ = square of the tangent from A to D.

$\therefore$ The point of contact of the tangent from A is a point on the circle A. Consequently the two circles A and D cut orthogonally.

Now, the circles of inversion are symmetrical; consequently the same relation must exist between any two of them.

Hence the circles cut each other orthogonally.

322. We have seen that the cyclic points on a quartic are also points on the circles of inversion. Therefore, the cyclic points on a curve are inverted into cyclic points on the inverse curve. For, at a cyclic point four consecutive points lie on a circle. When the curve is inverted, these four consecutive points are inverted into four consecutive points on the inverse curve and the circle is inverted into a circle through four consecutive points on the inverse curve. Hence it is a cyclic point. Since these points lie on the circles of inversion, all the above properties which we have proved for the foci are also true for the cyclic points.

323. *The locus of a point whose distances from three fixed points are connected by the relation $l\rho + m\rho' + n\rho'' = 0$, is a bicircular quartic having those three fixed points for foci.*

Take O, one of the fixed points, as origin and another point A situated on the axis of x at a distance c from the origin O. Let (x', y') be the co-ordinates of the third fixed point B.

Now, if P be the point (x, y) we have—

$$\rho^2 = x^2 + y^2 \; ; \; \rho'^2 = (x-c)^2 + y^2, \text{ and } \rho''^2 = (x-x')^2 + (y-y')^2.$$

$$\therefore \quad l\rho + m\rho' + n\rho'' = l\sqrt{x^2+y^2} + m\sqrt{(x-c)^2+y^2}$$

$$+ n\sqrt{(x-x')^2+(y-y')^2} = 0$$

or, $l\sqrt{u} + m\sqrt{v} + n\sqrt{w} = 0$, where u, v, w are point circles, whose centres are O, A and B respectively.

Rationalising the equation we obtain—

$$l^4u^2 + m^4v^2 + n^4w^2 - 2m^2n^2vw - 2n^2l^2wu - 2l^2m^2uv = 0 \quad \ldots \quad (1)$$

which is the equation of a bicircular quartic, when u, v, w are circles.

Now $u=0$ meets this quartic, where

$$m^4v^2 + n^4w^2 - 2m^2n^2vw = 0, \text{ i.e. } (m^2v - n^2w)^2 = 0.$$

Therefore $u=0$ touches the quartic at the two finite points, where $m^2v - n^2w = 0$ cuts it, *i.e.* the point circle $u=0$ has double contact with the bicircular quartic. Therefore the point O is a focus. Similarly, it can be shown that A and B are each a focus of the curve. We shall more fully discuss the properties of these curves in a subsequent chapter.

324. To determine the fourth focus of a bicircular quartic concyclic with three given foci.

The equation of a bicircular quartic having three fixed points as foci is $\quad l\sqrt{u} + m\sqrt{v} + n\sqrt{w} = 0. \quad \ldots \quad (1)$

To determine the fourth focus, concyclic with the three given ones, we consider the circle

$$\lambda u + \mu v + \nu w = 0 \quad \ldots \quad \ldots \quad (2)$$

This touches (1), if $\dfrac{l^2}{\lambda} + \dfrac{m^2}{\mu} + \dfrac{n^2}{\nu} = 0. \quad \ldots \quad (3)$

Hence the quartic (1) may be regarded as the envelope of (2), provided the condition (3) is satisfied. Now, (2) will be a point-circle if the discriminant vanishes, *i.e.* if

$$(\lambda+\mu+\nu)\{\mu c^2+\nu(x'^2+y'^2)\}-\nu^2y'^2-(c\mu-\nu x')^2=0 \quad \ldots \quad (4)$$

Now put $OB^2=b^2=x'^2+y'^2$ and $AB^2=a^2=(x'-c)^2+y'^2$.

$$\therefore\ 2cx'=a^2-b^2-c^2.$$

$\therefore$ The equation (4) becomes—

$$(\lambda+\mu+\nu)\{\mu c^2+\nu b^2\}-\nu^2b^2-\mu^2c^2-\mu\nu(b^2+c^2-a^2)=0,$$

or, $\mu\nu a^2+\lambda\nu b^2+\lambda\mu c^2=0$, *i.e.* $\dfrac{a^2}{\lambda}+\dfrac{b^2}{\mu}+\dfrac{c^2}{\nu}=0.$ ... (5)

Therefore, when the conditions (3) and (5) are satisfied, the equation (2) represents a point-circle having double contact with the curve. Hence, solving (3) and (5) we obtain the values of λ, μ, ν and therefore the fourth focus is determined.

325.* *If four concyclic foci of a bicircular quartic are given, through any point there can be described two such quartics and they cut each other at right angles.*

The equation of a bicircular quartic having three given foci is $l\sqrt{u}+m\sqrt{v}+n\sqrt{w}=0$ ... ... (1)

The fourth focus is determined by $\lambda u+\mu v+\nu w=0$,

where $\dfrac{l^2}{\lambda}+\dfrac{m^2}{\mu}+\dfrac{m^2}{\nu}=0$ ... (2) $\dfrac{a^2}{\lambda}+\dfrac{b^2}{\mu}+\dfrac{c^2}{\nu}=0$ ... (3)

When the fourth focus is given, the values of λ, μ, ν are given also. Hence, if the curve passes through any point (x', y', z'), we have

$$l\sqrt{u'}+m\sqrt{v'}+n\sqrt{w'}=0 \text{ and } \frac{l^2}{\lambda}+\frac{m^2}{\mu}+\frac{n^2}{\nu}=0.$$

* Salmon —H. P. Curves § 277.

But these two equations determine two sets of values of l, m, n, when (x', y', z') is fixed. Hence, two curves can be described through any given point.

Let $l\sqrt{u}+m\sqrt{v}+n\sqrt{w}=0$ and $l'\sqrt{u}+m'\sqrt{v}+n'\sqrt{w}=0$ be the equations of these two quartics. Since they are confocal, *i.e.* have four foci common, we have—

$$\frac{a^2}{\lambda}+\frac{b^2}{\mu}+\frac{c^2}{\nu}=0, \frac{l^2}{\lambda}+\frac{m^2}{\mu}+\frac{n^2}{\nu}=0, \frac{l'^2}{\lambda}+\frac{m'^2}{\mu}+\frac{n'^2}{\nu}=0.$$

Eliminating λ, μ, ν between these three equations, we obtain

$$a^2(m^2n'^2-m'^2n^2)+b^2(n^2l'^2-n'^2l^2)+c^2(l^2m'^2-l'^2m^2)=0..(4)$$

which is the condition that the two curves may be confocal.

In order to prove that the two curves cut each other orthogonally, it will be sufficient to find the condition that the generating circles of the two curves, namely,

$\lambda u+\mu v+\nu w=0$ and $\lambda' u+\mu' v+\nu' w=0$ cut each other orthogonally. This condition is found, after calculation, to be

$$a^2(\mu\nu'+\mu'\nu)+b^2(\nu\lambda'+\nu'\lambda)+c^2(\lambda\mu'+\lambda'\mu)=0. \quad .... \quad (5)$$

Now $\lambda u+\mu v+\nu w=0$ touches the quartic

$$l\sqrt{u}+m\sqrt{v}+n\sqrt{w}=0, \text{ if } \frac{l^2}{\lambda}+\frac{m^2}{\mu}+\frac{n^2}{\nu}=0.$$

$\therefore$ if we take $\lambda : \mu : \nu = \frac{l}{\rho} : \frac{m}{\rho'} : \frac{n}{\rho''}$,

the circle $$\frac{l}{\rho}u+\frac{m}{\rho'}v+\frac{n}{\rho''}w=0 \quad ... \quad (6)$$

touches the quartic at the point where, $\sqrt{u}$, $\sqrt{v}$, $\sqrt{w}$ are ρ, ρ', ρ''.

Similarly, $\frac{l'}{\rho}u+\frac{m'}{\rho'}v+\frac{n'}{\rho''}w=0$ touches the second quartic $l'\sqrt{u}+m'\sqrt{v}+n'\sqrt{w}=0$ at the same point. These two circles cut orthogonally, if

$$a^2\cdot\frac{mn'+m'n}{\rho'\rho''}+b^2\frac{nl'+n'l}{\rho\rho''}+c^2\frac{lm'+l'm}{\rho\rho'}=0. \quad ... \quad (7)$$

But solving the equations $l\rho+m\rho'+n\rho''=0$ and $l'\rho+m'\rho'+n'\rho''=0$, for ρ, ρ', ρ'', we obtain

$$\rho:\rho':\rho''=mn'-m'n:\ nl'-n'l:\ lm'-l'm.$$

Substituting these values in (7), we find—

$$a^2(m^2n'^2-m'^2n^2)+b^2(n^2l'^2-n'^2l^2)+c^2(l^2m'^2-l'^2m^2)=0,$$

which is the same condition that the two curves are confocal. Hence the truth of the theorem follows.

326. Bitangents of bicircular quartics:

The equation of a quartic is $UW=V^2$, where U, V, W are conics. The generating conic is $\lambda^2U+2\lambda V+W=0$.

We have seen that there are six values of λ for which this represents two right lines and these right lines are bitangents to the quartic. Now, in the case of binodal quartics, U, V, W, and consequently $\lambda^2U+2\lambda V+W$ all pass through the two nodes. Hence, when $\lambda^2V+2\lambda V+W$ denotes two right lines, it denotes two lines passing one through each of the nodes, or else, the line through the nodes and another line. In the first case, the lines are not proper bitangents, but ordinary tangents passing through the nodes. In the second case, one is a proper bitangent and the other is the line joining the nodes. Hence, a binodal quartic has only two proper bitangents corresponding to two values of λ. For, if U, V, W have two common points, the equation of V and W will be of the forms $kU+LM$ and $k'U+LN$; and $\lambda^2U+2\lambda V+W$ becomes

$$(\lambda^2+2k\lambda+k')U+L(2\lambda M+N)=0.$$

Therefore, $\lambda^2U+2\lambda V+W$ will have L for a factor, if $\lambda^2+2\lambda k+k'=0$. The two values of λ given by this equation will give us two bitangents of the form $2\lambda M+N=0$.

When U, V, W are all circles, the curve has two nodes at the circular points at infinity and it is a bicircular quartic.

In this case $\lambda^2 U + 2\lambda V + W = 0$ becomes of the form

$$(a\lambda^2 + 2b\lambda + c)(x^2 + y^2) + \text{ linear terms} = 0.$$

Hence, if this is to represent two lines one of which is the line at infinity, we must have the coefficient of $x^2 + y^2 = 0$,

$$i.e.\ a\lambda^2 + 2b\lambda + c = 0,$$

which gives two values of λ. Corresponding to each value, we obtain one proper bitangent to the curve, whose equation is $2\lambda M + N = 0$, which always passes through the point $M = N = 0$, *i.e.* the radical centre of U, V, W.

Now, there are four ways of reducing the equation of a given bicircular quartic to the form $UW = V^2$. Corresponding to each, there are two proper bitangents to the curve. Hence, *a bicircular quartic has eight proper bitangents.*

327. The samething follows also from the equation

$$4a^2(x+f)^2 + 4b^2(y+g)^2 = (x^2 + y^2 - c)^2.$$

For, the equation can be written as—

$$\{(2a(x+f) + i2b(y+g)\}\ \{2a(x+f) - i2b(y+g)\}$$
$$= (x^2 + y^2 - c)^2,$$

which shows that the two lines $2a(x+f) + 2ib(y+g) = 0$ and $2a(x+f) - 2ib(y+g) = 0$ touch the quartic at the two points, where each intersects the circle $x^2 + y^2 - c = 0$. They are, therefore, bitangents of the curve. These are real or imaginary, according as the focal conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is a hyperbola or an ellipse.

These two bitangents intersect at the point $x = -f$, $y = -g$, which is the centre of inversion.

Hence, *through a centre of inversion there pass two bitangents of the curve*; and since there are four such centres

of inversion, the number of bitangents of a bicircular quartic is eight.

328. Bicircular quartics can be divided into two classes, according as the curve has two or three nodes. The case of two nodes we have already discussed. Now we shall study some properties of a bicircular quartic having a third node in the finite part of the plane.

The inverse of a conic with respect to any point, not on the curve, is a bicircular quartic having a third double point at the centre of inversion; and this point will be an acnode, a cusp or a crunode, according as the conic is an ellipse, parabola or a hyperbola.

The equation of a conic, referred to any point as origin, is
$$u_2+u_1+u_0=0.$$

The inverse of this curve is—
$$u_0(x^2+y^2)^2+k^2u_1(x^2+y^2)+k^4u_2=0,$$
which is evidently a bicircular quartic. The equation contains no constant or linear terms, and therefore the origin is a double point. It will be a node, a cusp or a conjugate point, according as u_2 represents a pair of lines, (1) real and distinct, (2) coincident, or (3) imaginary; *i.e.* according as the given conic is a hyperbola, parabola or an ellipse.

When the origin lies on the conic, $u_0=0$, and the inverse curve reduces to a circular cubic.

329. *The pedal of a central conic with respect to any point in its plane is a bicircular quartic, having a third double point at the origin, which is a node, a cusp or a conjugate point, according as the origin lies without, on or within the conic.*

Let the equation of a conic be—
$$ax^2+by^2+2hxy+2fy+2gx+c=0 \qquad \dots \quad (1)$$

The condition that any line $x\cos\theta+y\sin\theta=p$ touches this may be written as—

$$A\cos^2\theta+B\sin^2\theta+Cp^2-2F\sin\theta.p-2G\cos\theta.p$$
$$+2H\sin\theta\cos\theta=0 \quad (2)$$

where A, B, C,.........are the minors of the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \equiv \Delta$$

The coordinates of the foot of the perpendicular are $p\cos\theta$ and $p\sin\theta$. Multiplying (2) by p^2 and substituting x and y for $p\cos\theta, p\sin\theta$ and $p^2=x^2+y^2$, we obtain

$$Ax^2+By^2+C(x^2+y^2)^2-(2Fy+2Gx)(x^2+y^2)$$
$$+2Hxy=0,$$

i.e. $$C(x^2+y^2)^2-2(Fy+Gx)(x^2+y^2)$$
$$+(Ax^2+By^2+2Hxy)=0,$$

which is a bicircular quartic, having a double point at the origin.

Now, the tangents at the origin are given by—

$$Ax^2+2Hxy+By^2=0,$$

and these are real and distinct, coincident or imaginary, according as $AB-H^2 \lesseqgtr 0$, *i.e.* according as $\Delta.c \lesseqgtr 0$ (3)

Now, when the origin lies outside the conic, two real tangents can be drawn to it and the area of the triangle formed by these tangents and their chord of contact is real. When the point lies on the conic, the area vanishes, and when it lies inside the conic, it is imaginary.

The expression for area* is—

$$\frac{c\sqrt{-\Delta.c}}{\{(ab-h^2)c-\Delta\}}$$

* A Memoir on Plane Analytical Geometry by Sir Asutosh Mukhopadhyay, Kt., D.Sc., Ph.D.,—Journal of the Asiatic Society of Bengal, Vol. LVI, part II, No. 3 (1887), § 21.

which is real, zero, or imaginary, according as $\Delta . c \lesseqgtr 0$, which is the same condition as (3).

Hence the conclusion follows.

Cor: If $C=0$, *i.e.* $ab-h^2=0$, the curve reduces to a circular cubic, but in this case the given conic becomes a parabola. Hence, *the pedal of a parabola is a circular cubic.*

330. *If a circle of inversion of a bicircular quartic touches the corresponding focal conic, the point of contact is a node on the quartic; when it osculates the focal conic, the point of contact is a cusp.*

When the circle of inversion touches the focal conic, it meets it in two consecutive points. Therefore, the variable circle in two consecutive positions pass through the same point. Consequently the two tangents to the circles in consecutive positions pass through the same point on the quartic, *i.e.* the line joining the centre of inversion with the point of contact cuts the quartic in two coincident points. Hence the point of contact is a node on the quartic.

Again, if the focal conic touch the Jacobian circle or the circle of inversion in two points, each of them will be a node, *i.e.* the quartic will break up into two circles each passing through the two nodes.

Otherwise:

We may prove this analytically as follows :—

Let $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ be the focal conic and

$x^2+y^2+2fx+2gy+c=0$ be the circle of inversion. Then the equation of the bicircular quartic referred to O, the centre of the focal conic, is—

$$4a^2(x+f)^2+4b^2(y+g)^2=(x^2+y^2-c)^2 \quad \ldots \quad (1)$$

Let the circle of inversion touch the focal conic at P, whose coordinates are $(a\cos\alpha,\ b\sin\alpha)$.

Now, the tangent at P to the focal conic is—

$$\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}=1 \quad \ldots \quad \ldots \quad (2)$$

Comparing this with the equation $x\cos\theta+y\sin\theta=p$, where p is the perpendicular on the tangent from O, we find

$$\frac{\cos\alpha}{a\cos\theta}=\frac{\sin\alpha}{b\sin\theta}=\frac{1}{p}. \quad \ldots \quad \ldots \quad (3)$$

Also, we have AP parallel to p;

$\therefore \dfrac{+f+a\cos\alpha}{\cos\theta}=\dfrac{+g+b\sin\alpha}{\sin\theta}=\delta$, where δ is the radius of the circle. $\therefore$ $f+a\cos\alpha=\delta\cos\theta$, $g+b\sin\alpha=\delta\sin\theta$.

From (3), $ap\cos\alpha=a^2\cos\theta$, $bp\sin\alpha=b^2\sin\theta$.

and $p^2=a^2\cos^2\theta+b^2\sin^2\theta$.

Transfering the origin to P, the equation of the quartic (1) becomes—

$$4a^2(x+f+a\cos\alpha)^2+4b^2(y+g+b\sin\alpha)^2$$
$$=\{(x+a\cos\alpha)^2+(y+b\sin\alpha)^2-c\}^2$$

or, $4a\,(x+\delta\cos\theta)^2+4b^2(y+\delta\sin\theta)^2$

$$=\{x^2+2ax\cos\alpha+a^2\cos^2\alpha+y^2+b^2\sin^2\alpha+2by\sin\alpha-c\}^2$$

or, $4a^2(x^2+\delta^2\cos^2\theta+2x\delta\cos\theta)$

$$+4b^2(y^2+\delta^2\sin^2\theta+2\delta y\sin\theta)$$

$$=\{x^2+y^2+2ax\cos\alpha+2by\sin\alpha+a^2\cos^2\alpha+b^2\sin^2\alpha-c\}^2$$

or, $4a^2x^2+4b^2y^2+4\delta^2(a^2\cos^2\theta+b^2\sin^2\theta)$

$$+8\delta(a^2\cos\theta x+b^2\sin\theta y)$$

$$=\{r^2+2ax\cos\alpha+2by\sin\alpha+(\delta\cos\theta-f)^2$$
$$+(\delta\sin\theta-g)^2-c\}^2$$

$$=\{r^2+2ax\cos\alpha+2by\sin\alpha+\delta^2-2f\delta\cos\theta$$
$$-2g\delta\sin\theta+f^2+g^2-c\}^2$$

$$=\{r^2+2ax\cos\alpha+2by\sin\alpha+2\delta(\delta-f\cos\theta-g\sin\theta)\}^2$$

$$\therefore\ 4a^2x^2+4b^2y^2+4p^2\delta^2+2p\delta(ax\cos\alpha+by\sin\alpha)$$
$$=\{r^2+2ax\cos\alpha+2by\sin\alpha+2\delta p\}^2,$$
$$\text{for, } \delta-f\cos\theta-g\sin\theta=p.$$

Thus the above equation, referred to P as origin, wants the constant and the linear terms. Hence the origin P is a double point, which is a node, a cusp or a conjugate point, as will be found on reduction, according as

$$a^2b^2\sin^2\alpha\cos^2\alpha \gtreqless (a^2\cos^2\alpha+p\delta-a^2)(b^2\sin^2\alpha+p\delta-b^2),$$

$$i.e.\ a^2\sin^2\alpha+b^2\cos^2\alpha \gtreqless p\delta,$$

which reduces to $OP'^2 \gtreqless p\delta$, where OP′ is the diameter conjugate to OP.

When $OP'^2=p\delta$, δ becomes the radius of curvature at P. Therefore, the point of contact P will be a cusp, when the circle of inversion osculates the focal conic.

The quartic has two imaginary nodes at the circular points and therefore the third node must be real. Again, when the circle of inversion touches the focal conic, two foci coincide at the point of contact, and these foci must be real. In the case of a cusp, three real single foci coincide at the cusp. When the circle of inversion has double contact with the focal conic, the quartic has two additional nodes and the curve, therefore, must be degenerate, consisting of two circles passing through the two additional nodes.

When the focal conic touches the line at infinity (the line forming a part of the Jacobian), that line forms a part of the quartic. Thus, if the focal conic be a parabola, the bicircular quartic reduces to a circular cubic together with the line at infinity.

CHAPTER XIX.

Circular cubics as degenerate bicircular quartics.

331. We have seen that a binodal quartic reduces to a cubic and the line joining the nodes, when the focal conic touches this line. In the case of a bicircular quartic, the line joining the nodes is the line at infinity, and consequently, when the focal conic becomes a parabola, the bicircular quartic reduces to a circular cubic and the line at infinity.

Let $y^2=4ax$ (1) be the focal conic and $x^2+y^2+2fx+2gy+c=0$ (2) be the circle of inversion.

Any point on the focal parabola can be taken as $(at^2, 2at)$. Thus the equation of a circle having the centre at $(at^2, 2at)$ is

$$(x-at^2)^2+(y-2at)^2=r^2 \quad \ldots \quad \ldots \quad (3)$$

If (3) cuts (2) orthogonally, we must have—

$$(at^2+f)^2+(2at+g)^2=r^2+f^2+g^2-c$$

$$\text{or, } r^2=a^2t^4+(2af+4a^2)t^2+4agt+c.$$

$\therefore$ The equation of the variable circle becomes—

$$(x-at^2)^2+(y-2at)^2=a^2t^4+(2af+4a^2)t^2+4agt+c$$

$$\text{or, } (2af+2ax)t^2+4a(y+g)t=x^2+y^2-c.$$

$$\text{i.e., } 2a(x+f)t^2+4a(y+g)t=x^2+y^2-c. \quad \ldots \quad (4)$$

Differentiating this with respect to t, we get

$$4a(x+f)t+4a(y+g)=0$$

$$\text{or, } t=-\frac{y+g}{x+f}. \quad \ldots \quad \ldots \quad (5)$$

$\therefore$ The envelope of (4) becomes—

$$2a\,\frac{(y+g)^2}{(x+f)}-4a.\,\frac{(y+g)^2}{(x+f)}=x^2+y^2-c$$

$$\text{or,}\ (x^2+y^2-c)(x+f)=-2a(y+g)^2$$

$$\text{or,}\ (x^2+y^2-c)(x+f)+2a(y+g)^2=0.\quad \ldots \quad (6)$$

This is the equation of a circular cubic. All the properties proved for a bicircular quartic must hold for a circular cubic, which is only a degenerate form of the former.

We have seen that there are four circles of inversion and four corresponding focal conics associated with any bicircular quartic. These focal conics are confocal and therefore they have the same real foci and axis. Consequently, when one focal conic is a parabola, the other three must be parabolas, and the circular cubic can be generated by taking any of them as focal conic.

332. The equation (6) of the cubic shows that the curve passes through the two circular points at infinity and through another point at infinity in the direction of the line $x+f=0$, which must be a real point. Again, $x+f=0$ touches the curve where $(y+g)^2=0$, *i.e.* at the point $x=-f$, $y=-g$, *i.e.* the line $x+f=0$ is a tangent to the curve at the centre of inversion $(-f, -g)$. But $x+f=0$ gives the direction of the real asymptote to the cubic. Therefore the tangent $x+f=0$ meets the cubic again in a real point at infinity. Thus the four centres of inversion lie on the four tangents drawn to the curve from the real point at infinity. These tangents are therefore parallel to the real asymptote.

333. We have shown that the generating circle has double contact with the bicircular quartic, the chord of

contact passing through a centre of inversion. The same thing holds in the case of a circular cubic;

or, in other words, *if a line through a centre of inversion intersects the circular cubic in two points, a circle can be drawn having simple contact with the cubic at these two points.*

For, let A be a centre of inversion and any line through A intersect the cubic in two other points B and C. Now, we have seen that A is the point of contact of a tangent drawn from the real point K at infinity on the curve. Consider a line A′B′C′ consecutive to ABC. (Fig. 38).

We have the following three cubics passing through the eight points A,B,C, A′,B′,C′, K and I :—

(1) The given cubic ;

(2) The cubic consisting of the three lines ABC, A′B′C′ and KIJ.

(3) The line AA′K and a conic determined by B, C, B′, C′, I.

Therefore they must pass through a common ninth point J.

Hence, the six points B, C, B′, C′, I, J lie on a conic, which must be a circle ; and when ABC, A′B′C′ coincide, the circle has double contact with the cubic at B and C.

334. We have shown that the four centres of inversion of a bicircular quartic are such that each is the orthocentre of the triangle determined by the other three and the triangle is self-conjugate with respect to the circle, whose centre is the point. This is also true in the case of a circular cubic, but in this case, the four centres lie on the cubic. Now we shall show that, of the four circles of inversion, one is imaginary and the other three real.

Let D be the centre of the circle with respect to which the triangle ABC is self-conjugate, so that A is the pole of BC. Let AD meet BC in E.

Then $DA.DE=r^2$, where r is the radius of the circle.

Now, when A and E lie on the same side of the centre D, DA and DE are of the same sign and consequently DA.DE is positive. But if A and E are on opposite sides of D, DA and DE are of opposite signs and consequently DA.DE is a negative quantity, and therefore the radius r is imaginary. (Fig. 39.)

Now, it is seen that of the four triangles formed by ABCD, one is acute-angled and the other three obtuse-angled, and in the case of an acute-angled triangle the radius becomes imaginary. Hence, of the four circles of inversion one is imaginary and the other three real.

If we invert the cubic with respect to a real circle of inversion whose centre is A (say), then the inverse of D will be a point on the cubic. For, since BCD is self-conjugate with respect to the circle A, $AD.AE=r^2$, and E is the inverse of D, but the cubic is inverted into itself. Hence the inverse E of D lies also on the cubic. Similarly, if F and G are the feet of the perpendiculars BD and CD, F and G lie on the cubic. [But this is not true for the bicircular quartic.]

Thus we have the theorem :—Every circular cubic passes through the four centres of inversion and also through the feet of the perpendiculars of the triangle formed by joining any three centres of inversion.

335. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the four concyclic foci of the circular cubic. Then the focal parabola intersects the circle of inversion in the four points $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. Hence, by a known theorem in geometry, the chords $\alpha_1\alpha_3$

and $\alpha_2\alpha_4$ are equally inclined to the axis of the parabola. Hence the axis bisects the angle between the two chords. Since there are two bisectors, we obtain two parabolas in the pencil of conics through $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, whose axes are mutually orthogonal. Hence, given four concyclic foci, there are two circular cubics, the axes of whose focal parabolas are mutually orthogonal.

Again, we have seen that the diagonal points of the quadrangle $\alpha_1\alpha_2\alpha_3\alpha_4$ are three centres of inversion of the bicircular quartic. The same must be true also for the circular cubic. Let A, B, C be these diagonal points. Then the fourth centre of inversion, *i. e.* the centre of the circle through $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ is the orthocentre D of ABC.

Now, if we take ABC for the triangle of reference, the equation of a conic through $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ may be written as $lx^2+my^2+nz^2=0$. (1) If this is to be a parabola, the condition is $a^2/l+b^2/m+c^2/n=0$, (2) where a, b, c are the lengths of the sides of the triangle ABC. Now, let E, F, G be the mid-points of the sides of the triangle ABC. Then the equation of the line EF is $by+cz-ax=0$, which evidently touches the parabola, in virtue of the relation (2). Similarly, FG and EG touch the same parabola. Hence the focal parabola is inscribed in the triangle EFG. Therefore its focus lies on the circle circumscribing the triangle EFG. Now, the circle circumscribing EFG is the nine-points circle of the triangle ABC. Hence, the focus of the focal parabola lies on the nine-points circle of the triangle formed by the three centres of inversion. But the four triangles formed by the four centres A, B, C, D of inversion, taken three at a time, have a common nine-points circle. Hence we obtain the theorem :—

The foci of the focal parabolas lie on the common nine-points circle of the triangles formed by any three of the four centres of inversion.

336. Again, we have shown that the focal parabola touches the sides of the triangle formed by joining the mid-points E, F, G of the triangle ABC. Hence the directrix of the parabola passes through the orthocentre of the triangle EFG.* But the orthocentre of EFG is the circumcentre of the triangle ABC. Hence the directrix passes through the circumcentre of the triangle ABC. Thus we obtain the theorem :—

The directrices of the focal parabolas respectively pass through the centres of the circles circumscribing the triangles formed by the centres of inversion, taken three at a time.

337. We have shown that the equation of a circular cubic in Trilinear co-ordinates can be written in the form $u_1 C = IS$, where C is a circle, S any conic and $I=0$ is the line at infinity.

The form of the equation shows that—

(*i*) the cubic passes through the two circular points at infinity.

(*ii*) it passes through the point where u_1 intersects the line at infinity, *i.e.* u_1 is parrallel to the real asymptote of the curve.

(*iii*) it passes through the points where the conic S intersects the circle C and the line u_1.

(*iv*) the circle C intersects the cubic in two points at I and J, and in four finite points.

338. We shall now prove the following important theorem :—If a circle intersects a circular cubic in four

* Salmon—Conics § 297, *Ex.* 14.

points, then the opposite sides of the quadrangle formed by these four points intersect the cubic again in three pairs of points. The line joining each pair of these points is parallel to the asymptote.

Let the circle intersect the cubic in four points A, B, C, D. Let the pairs of opposite sides AB, CD ; AC, BD ; BC, AD intersect the cubic again in the three pairs of points E, F ; E′, F′ ; E″,F″ respectively. Then the lines EF, E′F′, E″F″ are parallel to the asymptote of the cubic.

Let S be the equation of the circle and $U+\lambda V=0$ be the equation of a conic through the four points A, B, C, D. Then the equation of a system of circular cubics through A, B, C, D can be written as $u_1 S + I(U+\lambda V)=0$, where u_1 is a line parallel to the asymptote and each individual member is determined by giving a definite value to λ.

Now, the equation shows that the cubic passes through the points where u_1 intersects $U+\lambda V=0$. If we determine λ so that $U+\lambda V=0$ represents two right lines, the curve passes through the points where u_1 intersects those two lines. Again, there are three values of λ for which $U+\lambda V$ breaks up into two right lines; corresponding to each value we get a definite curve. Now, if E and F are the points where AB and CD, the lines constituting the conic $U+\lambda V$, intersect the cubic again, u_1 must be the line EF and it is parallel to the asymptote. Similarly, E′F′, E″F″ are also parallel to the asymptote.

Cor. 1. If A and B coincide, the line AE becomes the tangent at A and we obtain the theorem :—

If a circle touch a circular cubic at A and intersect it in two points C and D, then the line joining the tangential point of A and the third point where CD cuts the cubic again is parallel to the asymptote.

Cor. 2. If A, B as well as C, D coincide, we have :—

If a circle have simple contact with a circular cubic at two points, the line joining the tangentials of the two points of contact is parallel to the asymptote and the chord of contact intersects the curve at a third point the tangent at which is parallel to the asymptote, *i.e.* the point is a centre of inversion.

Cor. 3. If the three points A, B, C coincide, then the circle becomes the circle of curvature at A and the line AD is the chord of curvature. Hence the theorem becomes :—

If the circle of curvature at a point A of a circular cubic intersects the curve again in D, the line joining the tangential of A and the third point where the chord of curvature intersects the cubic again is parallel to the asymptote.

Cor. 4. If the four points A, B, C, D coincide, the point A becomes a cyclic point and the theorem becomes :—

The tangent at a cyclic point of a circular cubic intersects the cubic again in a centre of inversion. For, in this case the tangent at the tangential of the cyclic point, is parallel to the asymptote. This point, therefore, must be a centre of inversion, as we have already shown. Thus, the cyclic points on the cubic are the points of contact of the tangents drawn to the curve from the centres of inversion and consequently there are sixteen cyclic points.

339. We have shown that the inverse of a bi-circular quartic is a circular cubic, when the origin is a point on the curve. Conversely, when a circular cubic is inverted from any point not on the curve, the inverse

is a bicircular quartic, but if the origin is a point on the curve, it inverts into another circular cubic.

Let $u_1 r^2 + v_2 + v_1 + v_0 = 0$ be the equation of a circular cubic. The inverse of this with respect to the origin is

$$\frac{k^2}{r^2}u_1 \cdot \frac{k^4}{r^4} \cdot r^2 + \frac{k^4}{r^4}v_2 + \frac{k^2}{r^2}v_1 + v_0 = 0,$$

or, $v_0 r^4 + k^2 r^2 v_1 + k^4(v_2 + k^2 u_1) = 0$, which passes through the origin and is a bicircular quartic.

If the cubic passes through the origin, $v_0 = 0$ and the inverse curve becomes $r^2 v_1 + k^2 v_2 + k^4 u_1 = 0$, which is another circular cubic through the origin; and the two curves are such that the tangent to one at the origin is parallel to the asymptote of the other.

Again, if the line $u_1 = 0$ is the asymptote, then u_1 must be a factor of v_2, so that $v_2 = u_1 u_1'$. Then the inverse curve is $r^2 v_1 + k^2 u_1 u_1' + k^4 u_1 = 0$, which shows that the origin is a point of inflexion on the inverse cubic. Hence we have the theorem :—*If a circular cubic be inverted from the point where the asymptote cuts the curve, the point will be a point of inflexion on the inverse curve and the asymptote will invert into the inflexional tangent.*

This also follows from the following geometrical consideration :—

Since the asymptote passes through the origin, its inverse will be a line through the origin, and the origin inverts into itself. The two consecutive points of contact at infinity of the asymptote invert into two consecutive points at the origin on the inverse of the asymptote. Hence, three consecutive points at the origin lie on a right line, which is therefore a point of inflexion.

340. From the above theorem we at once deduce the following by inversion :—*The points of contact of the tangents which can be drawn to a circular cubic from the point where it is met by the asymptote lie on a circle,* which is the nine-points circle of the four triangles formed by the centres of inversion.

Let O be a point of inflexion on a circular cubic. Then the points of contact of the three tangents, which can be drawn from O, lie on the harmonic polar of O. If we invert the curve with respect to O, the inverse curve will be a circular cubic, the inflexional tangent will invert into the asymptote which passes through O. The tangents from O will invert into tangents from the same point to the inverse curve, while the harmonic polar inverts into a circle through O. Hence the truth of the theorem follows.

CHAPTER XX.

SPECIAL QUARTIC CURVES.

341. In the present chapter we shall discuss the properties of a variety of well-known curves of the fourth degree, which have acquired historical importance, being associated for the most part with the names of some of the greatest geometers of past ages.

The Oval of Cassini.

342. The Oval of Cassini, known as the Cassinian, is the locus of a point which moves so that the product of its distances from two fixed points is constant. (Fig. 40).

The Equation of the Cassinian.

Let F_1 and F_2 be the two fixed points and let ρ and ρ' be the distances of any point from F_1 and F_2 respectively. The bi-polar equation of the locus is

$$\rho\rho' = k^2 = a \ const. \quad \ldots \quad \ldots \quad (1)$$

To obtain the Cartesian equation, we take the middle point O of F_1F_2 as the origin and OF_1 as the axis of x. Let $F_1F_2 = 2a$.

If P be the point (x, y), then we have—

$$F_1P^2 = (x+a)^2 + y^2 \text{ and } F_2P^2 = (x-a)^2 + y^2.$$

But $F_1P^2 . F_2P^2 = \{(x+a)^2 + y^2\}\{(x-a)^2 + y^2\}$

Therefore $(x^2 + y^2 + a^2)^2 - 4a^2x^2 = k^4. \quad \ldots \quad (2)$

When $k^2 = c^2 - a^2$, it becomes—

$$(x^2 + y^2 + a^2) - 4a^2x^2 = (c^2 - a^2)^2.$$

This equation can again be written in polars as

$$r^4 + a^4 - 2r^2a^2\cos 2\theta = k^4. \quad \ldots \quad \ldots \quad (3)$$

The equation (2) shows that the Cassinian is a bicircular quartic.

If $k=a=c/\sqrt{2}$, the equation (3) reduces to

$$r^2=c^2\cos 2\theta,$$

which is called the Lemniscate of Bernoulli.

343. Properties of the Cassini's ovals:

1. The Cassinian always cuts the axis of x in the two real points $x=\pm\sqrt{k^2+a^2}$, *i.e.* two fixed points.

2. It consists of two detached ovals, each of which encloses one of the points F_1 and F_2 which are, as we shall show hereafter, the two triple foci, when $\sqrt{2}.a>c$.

3. It is a binodal quartic, having a pair of biflecnodes at the circular points.

Take the two circular lines $x\pm iy=0$ as two sides of the triangle of reference and the line at infinity as the third side.

Then the equation (2) reduces to the trilinear form

$$\square(\alpha\beta+a^2I^2)^2-a^2I^2(\alpha+\beta)^2=k^4I^4. \qquad \ldots \quad (4)$$

This form of the equation shows that the circular points are nodes (biflecnodes) * on the curve.

4. The Plücker's numbers are—

$$n=4,\quad m=8,\quad \delta=2,\quad k=0,\quad \tau=8,\quad i=12.$$

Since four points of inflexion are situated at the circular points, the curve has only eight other points of inflexion. Of these four must be imaginary and the remaining four may be all real or imaginary, or may coincide into two real points of undulation.

5. The two fixed points are triple foci of the curve. For, the nodal tangents at the circular points are

$$\beta^2=a^2I^2,\ \alpha^2=a^2I^2.$$

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* See § 272.

These in Cartesian co-ordinates become—

$$(x-iy)^2=a^2 \text{ and } (x+iy)^2=a^2.$$

and they intersect at the points $x=\pm a$, $y=0$, *i.e.* F_1 and F_2.

But these tangents are stationary tangents. Their points of intersection are therefore triple foci.

6. The Cassinian has two single foci.

For, a binodal quartic must have eight real foci, of which some may coalesce into multiple foci, and the two triple foci of a Cassinian are equivalent to six single foci.

The Cartesian Ovals.

344. If ρ and ρ' are the distances of any point from two given fixed points, then the loci defined by the equation $l\rho\pm m\rho'=n$ are called Cartesian Ovals.

This leads in general to an equation of the fourth degree in Cartesian co-ordinates. The following special cases are to be noticed :—

If $l=m$, $\rho\pm\rho'=$ *const.* gives an ellipse and a hyperbola, and if $n=0$, we obtain $\rho:\rho'=$ *constant*, which gives a circle. In fact, when any of the constants vanishes, it represents a circle. When $l=n$ or $m=n$, it is a limaçon.

In all cases, however, the normal at any point divides the angle between the radii vectores into two parts such that their sines are in the ratio $m:l$. For, we have

$l\dfrac{d\rho}{ds}\pm m\dfrac{d\rho'}{ds}=0$. The two fixed points F_1, F_2 are the two foci and there is a third real focus F_3.

It is easily seen that the curve consists of two ovals, one lying inside the other; the former corresponds to the equation $l\rho+m\rho'=n$, and the latter to $l\rho-m\rho'=n$.

345. We have seen that a bicuspidal quartic having two cusps at the circular points is called a Cartesian. Descartes studied this curve and it is known after him as the Oval of Descartes. Chasles* showed that a third point F_3 can be found on the line F_1F_2, whose distance ρ'' from the variable point P satisfies a relation of the from $l\rho \pm n\rho'' = c'$, *i.e.* the point F_3 is a third focus. In fact, a quartic having two cusps at I and J has three foci lying on a right line. When these foci are real, the curve is that studied by Descartes ; when two are imaginary, the curve is still called a Cartesian, though Descartes' mode of generation is no longer applicable. For a detailed account of this curve the student is referred to Prof. Williamson's Diff. Calc., Chap. XX, and Cayley's Memoir on Caustics. †

346. The Equation of the Cartesian :

Let P be the variable point and $F_1P = r$, $F_2P = r'$ $F_1F_2 = c$ and $\angle PF_1F_2 = \theta$.

Now, $$lF_1P \pm mF_2P = nc. \text{ (say)} \quad \ldots \quad (1)$$

$$\therefore\ l^2F_1P^2 + n^2c^2 - 2ncl\ F_1P = m^2F_2P^2.$$

But $$F_2P^2 = F_1P^2 + F_1F_2{}^2 - 2F_1P.F_1F_2 \cos\theta$$

$$= r^2 + c^2 - 2rc \cos\theta.$$

$$\therefore\ l^2r^2 + n^2c^2 - 2nclr = m^2(r^2 + c^2 - 2rc\cos\theta)$$

i.e. $$r^2\ (l^2 - m^2) - 2r(ncl - m^2c\cos\theta) + n^2c^2 - m^2c^2 = 0 \quad (2)$$

i.e. $$r^2(l^2 - m^2) - 2rc(nl - m^2\cos\theta) + c^2(n^2 - m^2) = 0,$$

or, in the simple form, $r^2 - 2(a + b\cos\theta)r + k^2 = 0 \ldots$ (3)

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* Chasles—Histoire de la Géométrie, note XXI, p. 352.

† Cayley—Coll. Papers, Vol. II, p, 336,

This is the polar equation of the curve, referred to F_1 as pole. Hence, any equation of the form

$r^2-2(a+b\cos\theta)r+c^2=0$ represents a Cartesian oval.

Prof. Williamson,* after giving the construction for the third focus F_3, collinear with F_1 and F_2, shows that the distances of any point on the outer oval from F_1 and F_3 are connected by an equation similar in form to (1). He then finds the equations of the curve relative to each pair of foci. Thus, if $F_1P=r_1$, $F_2P=r_2$, $F_3P=r_3$ and $F_1F_2=c_3$, $F_2F_3=c_1$, $F_3F_1=c_2$, then the equations of the curve may be written as :—

$mr_1\pm lr_2=nc_3$, referred to F_1 and F_2.

$nr_1\pm lr_3=mc_2$, referred to F_1 and F_3.

$nr_2-mr_3=\pm lc_1$, referred to F_2 and F_3.

If (x, y) be the coordinates of P, we have $x=r\cos\theta$; and the Cartesian equation of the curve becomes

$$\{r^2(l^2-m^2)+2cm^2x+(n^2-m^2)c^2\}^2=4n^2c^2l^2r^2 \quad \ldots \ (4)$$

or, changing the constants, it becomes of the form

$$\{x^2+y^2+2\lambda x+\mu\}^2+4k^2(px+q)=0.$$

The equation of the Cartesian may generally be brought to the form $S^2=k^3L$, where S represents a circle, L a right line, k being a constant, *i.e.* $k=0$ being the line at infinity.

This form of the equation shows that the intersections of S and k, *i.e.* the two circular points are cusps on the curve, the cuspidal tangents meeting at the centre of the circle S.

Consequently, the centre is the triple focus of the curve. The line L is a bitangent.

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* Diff. Calc. Chap. XX. §§303, 304.

347*. From the form of the above equation, it is easily seen that the Cartesian is the envelope of the variable circle $\lambda^2 kL + 2\lambda S + k^2 = 0$. The centre of this circle moves along a right line perpendicular to L. When the variable circle reduces to a point, we obtain a focus. Now, the discriminant of this equation is of the third degree in λ, which equated to zero gives three values corresponding to the three foci.

A Cartesian may also be generated as the locus of the vertex of a triangle whose base-angles move on two fixed circles, while the two sides pass through the centres of the circles, and the base passes through a fixed point on the line joining them.

Again, *If a circle cut a given circle orthogonally, while its centre moves along another given circle, its envelope is a Cartesian oval.*

This is given by Prof. Casey † as a particular case of a general and elegant property of bicircular quartics, namely, if the centre of the enveloping circle moves on a conic, instead of a circle, its envelope is a bicircular quartic. The three foci of the curve are the centre of the orthogonal circle, and the limiting points of the two fixed circles.

348.‡ Foci of Cartesians:

The equation of a Cartesian, referred to F_1, is

$$\{r^2(l^2-m^2)+2cm^2x+(n^2-m^2)c^2\}^2 = 4n^2c^2l^2r^2 \quad \dots (1)$$

The points, where the line $x+iy=0$ intersects it, are determined by the equation—

$$\{2cm^2x+(n^2-m^2)c^2\}^2=0,$$

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* This construction of a Cartesian is given by Salmon—H. P. Curves § 280.

† Casey—Transactions of the Royal Irish Academy 1869.

‡ *Cf.* A. B. Basset—loc. cit. §273.

which shows that this line is a tangent to the curve. Similarly, $x-iy=0$ is also a tangent. But these two lines intersect at the origin F_1, which is, therefore, a single focus. Again, the polar equation of the curve, referred to F_2 or F_3, is of the same form as (1); consequently those two points are also single foci and they are collinear.

Since the curve is a bicuspidal quartic of the sixth class, it has three single foci and one triple focus.

In order to determine the triple focus, we proceed as follows:—

Taking the circular lines through F_1 as the sides α and β of the triangle of reference and the line at infinity as the third side, the equation (1) reduces to the trilinear form—

$$\{\alpha\beta(l^2-m^2)+cm^2(\alpha+\beta)\mathrm{I}+(n^2-m^2)c^2\mathrm{I}^2\}^2=4n^2c^2l^2\alpha\beta\mathrm{I}^2.$$

The two cuspidal tangents are—

$$\left.\begin{aligned}\hat{\alpha}(l^2-m^2)+cm^2\mathrm{I}=0\\ \beta(l^2-m^2)+cm^2\mathrm{I}=0\end{aligned}\right\}$$

These equations in Cartesian coordinates become

$$(x\pm iy)\ (l^2-m^2)+cm^2=0.$$

They intersect at the real point $x=\dfrac{-cm^2}{l^2-m^2},\ y=0.$

which is, therefore, the triple focus.

It is evident that the three single foci and the triple focus all lie on the axis of x. Now, transfer the origin to the triple foc us and it will be found that the equation (4) of § 346 reduces to the form $\mathrm{S}^2+\mathrm{L}=0$, where S is a circle and L any right line.

Note: The circle S is called the focal circle and its centre is the triple focus. Since, by inversion, the focus of a curve is transformed into the focus of the inverse curve, and the origin is also a focus when I and J are cusps, the inverse of a Cartesian with respect to any point is a bicircular quartic, having three foci on a circle passing through the origin, which is also a focus.

349. *If any chord meet a Cartesian in four points, the sum of their distances from any focus is constant.*

The polar equation of a Cartesian, referred to any focus as pole, is $$r^2-2r\,(a+b\cos\theta)+c^2=0 \quad \ldots \quad (1)$$

The equation of any right line can be taken as—

$$r(l\cos\theta+m\sin\theta)=1 \quad \ldots \quad (2)$$

Eliminating θ between the equations (1) and (2), we shall obtain a biquadratic in r, in which the co-efficient of the second term, *i.e.* of r^3 is $-4a$; and consequently the sum of the roots is constant.

Note: If $c=0$, the equation becomes $r=a+b\cos\theta$, and therefore in addition to the two cusps at I and J, the curve has the origin for a node, and it is then called Pascal's *limaçon*. If further, $a=b$, the curve becomes tricuspidal and is called a *Cardioide.*

350. Points of inflexion :

A Cartesian has eight points of inflexion, which lie on a circular cubic.

From Plücker's formula, $i=3n(n-2)-6\delta-8\tau$, it follows at once that a Cartesian has only eight points of inflexion, it being a bicuspidal quartic.

Let the equation of the Cartesian be

$$r^2-2r(a+b\cos\theta)+c^2=0. \quad \ldots \quad (1)$$

Since the curvature at a point of inflexion vanishes and changes sign, the radius of curvature becomes infinite at that point. Hence the denominator of the expression

for radius of curvature, when equated to zero, gives the equation of the locus which passes through all the points of inflexion. If now we find $d^2u/\delta\theta^2$ from the above equation (1), and put $\delta^2 u/d\theta^2+u=0$,it is seen that the locus is a circular cubic.

Ex. Obtain the equation of the locus.

The Lemniscate of Bernoulli.

351. The Cartesian equation of this curve is—

$$(x^2+y^2)^2=a^2(x^2-y^2);$$

and the polar equation is $\quad r^2=a^2 \cos 2\theta.$

The form of the curve is that of a figure of eight. Evidently the origin is a biflecnode, with a pair of orthogonal tangents.

We have seen that the inverse of a rectangular hyperbola with respect to a vertex is the logocyclic curve. But, if the inverse is taken with respect to the centre, it is the Lemniscate of Bernoulli.

Let $x^2-y^2=a^2$ be the equation of a rectangular hyperbola. Its inverse *w.r.t.* the origin is—

$$a^4\left(\frac{x^2-y^2}{r^4}\right)=a^2, \qquad \text{or,} \qquad (x^2+y^2)^2=a^2(x^2-y^2).$$

Again, it is the pedal of the rectangular hyperbola *w.r.t.* its centre.

For, let X cos θ+Y sin $\theta=p$ be a tangent. Then, this must be the same as X.$2x$−Y.$2y$=Z.$2a$

$$\text{or X}.x-\text{Y}.y=a^2\text{Z}.$$

$$\therefore \quad \frac{x}{\cos\theta}=\frac{-y}{\sin\theta}=\frac{a^2}{p}.$$

$$\therefore \quad \left(\frac{a^2 \cos\theta}{p}\right)^2 - \left(\frac{-a^2 \sin\theta}{p}\right)^2 = a^2,$$

or, $\qquad a^2(\cos^2\theta - \sin^2\theta) = p^2.$

$\therefore$ The polar equation of the pedal is—

$$r^2 = a^2 \cos 2\theta,$$

which is the Lemniscate of Bernoulli.

We have seen that when $\sqrt{2}.\, a = c$, the Cassinian becomes the Lemniscate of Bernoulli. Therefore, all the properties of the Cassinian hold also for this curve.

The Limacon.

352. *The inverse of a conic with respect to its focus is a curve of the fourth order and is called a Limaçon.*

The polar equation of a conic, referred to its focus as pole, is $\qquad l/r = 1 - e\cos\theta.$

The inverse of this is $\rho = a - b\cos\theta$, putting $b = ae$.

If $a > b$, $e > 1$, the curve is the inverse of an ellipse and is called an *Elliptic Limaçon*. If $a < b$, $e < 1$, the curve is the inverse of a hyperbola and is called a *Hyperbolic Limaçon*. If $a = b$, $e = 1$, it is the inverse of a parabola and is called a Cardioid.

The Cartesian equation of the curve is given by—

$$(\rho^2 + b\rho\cos\theta)^2 = a^2\rho^2$$

i.e. $\quad (x^2+y^2)^2 + (x^2+y^2)(2bx - a^2) + b^2x^2 = 0,$

or, $(x^2+y^2+bx)^2 = a^2(x^2+y^2),$

which is evidently a bicircular quartic, having a double point at the origin, which is a conjugate point or a crunode, according as the limaçon is elliptic or hyperbolic.

Note: This curve was first studied by Pascal, who called it the Limaçon, from a fancied resemblance to the form of a snail. For a detailed account of the properties of this curve, the reader is referred to Basset's Cubics and Quartics, Chap. X, pp. 186-196.

353. *The Limaçon is the pedal of a circle with respect to any point in its plane.*

Let C be the centre of a circle and O any point in its plane. (Fig. 41.)

At any point A on the circle draw a tangent, and OP the perpendicular on the tangent. Let a be the radius of the circle, $OC=b$ and $\angle ACO=\theta$.

Then,
$$\begin{aligned} OP &= AC+OC \cos \angle POC \\ &= a-b\cos\theta. \end{aligned}$$

Therefore, the locus of P is $\rho=a-b\cos\theta$.

Hence, the locus is an elliptic or hyperbolic Limaçon, according as $a >$ or $< b$ *i.e.* according as O lies within or without the circle. When O lies on the circle, the locus is a Cardioid.

The limaçon can readily be traced by drawing from a fixed point on a circle any number of chords, and taking off a constant length on each of these chords, measured from the circumference of the circle.

The Cardioid.

354. We have seen that when $a=b$, the limaçon becomes a Cardioid. The Cartesian equation of the curve can be written as $(x^2+y^2)^2+2ax(x^2+y^2)=a^2y^2$.

It is the inverse of a parabola with respect to its focus and also the pedal of a circle with respect to a point on its circumference.

A Cardioid has one real cusp at the origin and two imaginary cusps at the circular points at infinity. It has one real bi-tangent and is of the third class.

The Conchoid of Nicomedes.

355. The Greek Geometer Nicomedes invented this curve for the purpose of trisecting an angle. The curve may be generated as follows :—

If through any fixed point O a secant POP′ be drawn meeting a fixed right line AB in V and on this two points P and P′ be taken such that VP=VP′=a constant, then the locus of P or P′ is called the Conchoid.

Draw OA perpendicular upon AB and let OA$=a$ and VP$=b$.

$\therefore$ Taking O for pole and the line OX parallel to AB for the initial line, the polar equation of the locus is

$$r=a \operatorname{cosec} \theta \pm b.$$

The curve consists of two branches, having the line AB for a common asymptote.

In the above equation, the + sign refers to the branch more remote from AB and the − sign refers to the branch nearer to AB. These two branches are called the *superior* and the *inferior* branch respectively. (Fig. 42)

The Cartesian equation of the curve is—

$(x^2+y^2)(y-a)^2=b^2y^2$, which includes both the branches.

The origin is a double point, the tangents being given by

$$a^2x^2+(a^2-b^2)\,y^2=0.$$

It is, therefore, a node, a cusp, or a conjugate point, according as $a \overset{<}{\underset{>}{=}} b$; *i.e.* OA $\overset{<}{\underset{>}{=}}$ VP.

The curve has a real tacnode at infinity. The curve passes through the circular points ; therefore, a circle which passes through the double point cannot intersect the curve in more than four points.

The curve was used for trisection of an angle and the insertion of two mean proportionals between two given straight lines.

356. Trisection of an angle :

Let the given angle be POM. (Fig. 43.) It is required to trisect the angle POM.

Let P be any point on OP and from Q, the middle point of OP, draw QM $(=a)$, perpendicular on OM.

Let $OQ=b$.

Through P draw the Conchoid $r=a \operatorname{cosec} \theta+b$, O being the origin and OM the initial line. With Q as centre and OQ radius, describe a circle cutting the other branch of the Conchoid in P′. Then OP′ trisects the angle POM.

Let $\angle POM=\theta$ and $\angle P'OM=\phi$.

Then, $OP'=a \operatorname{cosec} \phi-b$, since P′ lies on the lower branch of the curve. Now $QM=a=b \sin \theta$,

and $$OP'=OP \cos POP'=2b \cos(\theta-\phi)$$

$$\therefore \quad 2b\cos(\theta-\phi)=a\operatorname{cosec}\phi-b,$$

i.e. $2b\cos(\theta-\phi)\sin\phi=a-b\sin\phi=b\sin\theta-b\sin\phi$.

$$\therefore 2\cos(\theta-\phi)\sin\phi=\sin\theta-\sin$$

or, $$\sin(\theta-2\phi)=\sin$$

$$\therefore \qquad \theta=3\phi.$$

Consequently, OP′ trisects the angle POM.

The trisection of an angle can also be effected by means of the *Trisectrix*, which is a hyperbolic limaçon, when $b=2a$.

Note :—There is another class of curves called the *Roulettes*. It is not proposed to enter into a discussion of these curves in the present volume, but it will be taken up on a future occasion.

APPENDIX I.

NOTES ON THE BICIRCULAR QUARTIC.

Dr. Casey, in his remarkable Memoir on Bicircular Quartics, has given a number of interesting methods of generation of bicircular quartics. We reproduce below the most important of them :

The most general equation of a bicircular quartic may be written in the form

$$U \equiv (a, b, c, f, g, h,)(\alpha, \beta, \gamma)^2 = 0, \qquad \ldots \quad (1)$$

where $\alpha=0$, $\beta=0$, $\gamma=0$ are the equations of three given circles. By analogy to the trilinear system, the circles α, β, γ are called the *circles of reference.*

The equation $x\alpha+y\beta+z\gamma=0$ (2), where x, y, z are variable multiples, represents a circle. This will touch the locus (1), provided

$$\begin{vmatrix} a & h & g & x \\ h & b & f & y \\ g & f & c & z \\ x & y & z & 0 \end{vmatrix} = 0 \qquad \ldots \quad (3)$$

that is to say, the locus (1) is the envelope of the circle (2), provided the condition (3) is satisfied.

Now, if A, B, C be the centres of the three circles α, β, γ, it is evident that the centre O of the circle $x\alpha+y\beta+z\gamma=0$ is the mean centre of the points A, B, C. Also, the circle (1) is co-orthogonal with the three circles α, β, γ.

Let (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) be the coordinates of A, B, C respectively, and (X, Y, Z) those of the point O.

Then, $$X = x.x_1 + y.x_2 + z.x_3$$
$$Y = x.y_1 + y.y_2 + z.y_3$$
$$Z = x.z_1 + y.z_2 + z.z_3$$

Therefore, X, Y, Z are proportional to x, y, z, when X, Y, Z are regarded as areal coordinates of O, referred to ABC. Thus the locus of (X, Y, Z) is given by (3). Hence we obtain the theorem, as was otherwise obtained in §306 :—

If $(a, b, c, f, g, h)(\alpha, \beta, \gamma)^2 = 0$ *be the equation of a bicircular quartic, this quartic is the envelope of a variable circle, co-orthogonal with* α, β, γ, *whose centre moves along the conic whose equation in areal coordinates is the determinant* (3), *referred to the circles* α, β, γ.

The conic (3) is the focal conic of the bicircular quartic.

The equation $(a, b, c, f, g, h)(\alpha, \beta, \gamma)^2 = 0$ can be thrown into the form $SV_1 = kS_1V$, where k is a constant, and S, V, S_1, V_1 are four circles. Now, $SV_1 = kS_1V$ is the result of eliminating λ between the equations—

$$S + k\lambda S_1 = 0,$$
$$V + \lambda V_1 = 0.$$

Now, by varying λ, $S + k\lambda S_1$ gives a pencil of circles coaxal with S, S_1, and $V + \lambda V_1$ is a homographic pencil coaxal with V, V_1. Hence the following theorem is obtained :

A bicircular quartic may be described by the intersection of the homologous circles of two homographic pencils of circles.

From this, an equally simple method of describing a bicircular quartic is easily deduced :—

Consider two systems of coaxal circles through the points A, B and C, D respectively.

Institute a homographic relation between the circles of the two systems such that, to the circle with centre O corresponds the circle with centre O′; the centres E and E′ of the corresponding members are then determined so that $\angle EAO = \angle E'DO'$. Let these homologous circles intersect in the points P, P′. Then the locus of P and P′ is a bicircular quartic.

$$\text{Now,}\quad \angle APB - \angle CPD = \tfrac{1}{2}\angle AEB - \tfrac{1}{2}\angle CE'D$$
$$= \tfrac{1}{2}\angle AOB - \tfrac{1}{2}\angle CO'D$$
$$= \text{constant.}$$

Thus, the bicircular quartic may be regarded as the locus of the common vertex P of two triangles APB, CPD, such that the difference of their vertical angles is constant.

Hence we have the following theorem :

The locus of the common vertex of two triangles whose bases are given and whose vertical angles have a given difference is a bicircular quartic.

Similarly, when the sum of the vertical angles is given, the locus is a bicircular quartic.

Classification of Bicircular quartics :

Dr. Casey takes as the basis of his classification the species of the focal conic, which may be (1) an ellipse or a hyperbola, (2) a circle, (3) a parabola. Curves corresponding to each of those divisions have definite and marked distinctions, which he proceeds to enumerate in some details :—

(1) We have given in the body of the text the properties of such curves when the focal conic is an ellipse or a hyperbola.

(2) When the focal conic is a circle, the bicircular quartic becomes a Cartesian oval.

We have seen that the foci of the focal conic are the double foci of the bicircular quartic. Now, when the two foci coincide, the focal conic becomes a circle, and the tangents to the quartic at the circular points at infinity coincide, or, in other words, these points become cusps and the quartic becomes a Cartesian oval.

Hence, *a Cartesian oval is a bicircular quartic whose focal conics are circles.*

In the case of the Cartesian oval, the four confocal (focal) conics become three concentric circles and a right line through their centre.

The centres of inversion of a Cartesian oval are the foci of the curve.

Let the equations of the focal circle J and the circle of inversion F be $x^2+y^2=r^2$ and $(x+a)^2+y^2=r^2$ respectively. Then the perpendicular OT let fall from O, the centre of J, on a tangent to F is equal to $r-a\cos\theta$, where θ is the angle which OT makes with the axis of x. Then the points P, P′, points on OT, such that

$\mathrm{OT}^2-\mathrm{TP}^2=\mathrm{OT}^2-\mathrm{TP}'^2=k^2$, are points on the Cartesian oval. Denoting OP by ρ, we have—

$$k^2+\rho^2=\mathrm{OT}^2-\mathrm{TP}^2+\mathrm{OP}^2=\mathrm{OT}^2-\mathrm{TP}^2+(\mathrm{OT}+\mathrm{TP})^2$$
$$=2\mathrm{OT}.\mathrm{OP}.$$

i.e.
$$2(r-a\cos\theta)\rho=k^2+\rho^2$$

or,
$$2r\rho=k^2+x^2+y^2+2ax,$$

that is, $2r\rho=\mathrm{C}$, where C is a circle concentric with F.

$$\therefore 4r^2\rho^2=\mathrm{C}^2,\quad \text{or,}\quad 4r^2(x^2+y^2)=\mathrm{C}^2.$$

Hence, the circular lines $x\pm iy=0$ are tangents to the curve, so that the centre of the circle of inversion J is a focus.

This equation of a Cartesian oval shows that it is the envelope of the circle $(x^2+y^2)+\mu\mathrm{C}+\mu^2r^2=0$.

This circle has double contact with the oval. When $\mu = -1$, the circle becomes a right line, which shows that the oval has a double tangent. The expression for the radius of the circle is of the third degree in μ, and consequently, there are three values of μ which reduce it to a point-circle. Hence there are three collinear single foci of the curve.

(3) When the focal conic is a parabola, the quartic reduces to a circular cubic and the line at infinity. Since the focal conic is a parabola, some of the methods given before require modifications ; and, moreover, some methods are applicable to circular cubics, which have no analogue in bicircular quartics.

Thus, *the locus of the common vertex of two triangles whose bases are given, and whose vertical angles are equal, is a circular cubic.*

Dr. Casey has given a number of interesting properties of circular cubics, some of which we have given before.

Among others he gives the following :—

(1) *The osculating circles at the centres of inversion are the inverses of the asymptote with respect to the circles of inversion corresponding to these points.*

For, the inverse of the asymptote *w.r.t.* a centre of inversion is a circle through the origin and the two consecutive points at infinity on the asymptote invert into two consecutive points at the origin. Hence the asymptote inverts into the osculating circle.

(2) *The nine-points circle of the triangle formed by any three of the four centres of inversion of a circular cubic passes through the point where it meets its asymptote.*

Let D, E, F, be the feet of the perpendiculars of the triangle. Then the nine-points circle passes through these points.

Suppose the nine-points circle meets the cubic in a fourth point V, and O′ is the point where EF meets the cubic again; then DO′ is parallel to the asymptote* and if DV meets the cubic in O″, then O′O″ is parallel to the asymptote. Hence O″ coincides with D and DV is a tangent at D; consequently V is the point where it intersects the asymptote.

(3) *The tangents at D, E, F intersect at a point on the nine-points circle, where the asymptote meets the cubic.*

This follows very easily from (2).

From this he at once deduces the following theorem :—

(4) *The circle described through any three of the centres of inversion of a circular cubic meets the curve again where the cubic is met by the osculating circle at the fourth centre of inversion.*

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* For, A, B, C, O being the four centres of inversion, A, F, E, O lie on the cubic as well as on a circle, and hence by §338, O′D is parallel to the asymptote. In fact, if the sides DE, EF, FD, of the triangle DEF meet the cubic again in F′, D′, E′ respectively, then DD′, EE′, FF′ are parallel to the asymptote.

APPENDIX II.

A Note on Trinodal Quartics.

It is well-known that a Trinodal Quartic (Deficiency$=0$) has six points of inflexion. How are these six points situated ? Norman Ferrers* easily proved, years ago, that the six inflexional tangents envelope a conic, but left the *correlative* theorem that the six points lie on a conic unconsidered. The complete solution of this and cognate problems has, however, been given in an exhaustive Memoir† by A. Brill. The Memoir is worthy of careful study by students of analysis. Brill's treatment of the Trinodal Quartic may be briefly stated thus :—

The Trinodal Quartic

$$d_{11}x_2^2x_3^2+d_{22}x_3^2x_1^2+d_{33}x_1^2x_2^2$$
$$+2x_1x_2x_3(d_{23}x_1+d_{31}x_2+d_{12}x_3)=0$$

is inverted into the conic $d_{11}y_1^2+d_{22}y_2^2+\ldots=0$ by the quadric substitution :‡

$$y_1:y_2:y_3=x_2x_3:x_3x_1:x_1x_2=f_1:f_2:f_3.$$

(f_1, f_2, f_3), the point on the conic *corresponding* to (x_1, x_2, x_3) on the trinodal quartic, can, of course, in an infinite number of ways, be expressed as *quadratic* functions of a parameter.

Brill, then, avails himself of the Theory of Quadratic Forms§ and arrives at the *cubic* passing through the inflexional points :—

$$R.f_1.f_2.f_3+4V_1.V_{23}.V_{31}=0\ldots\ldots\ldots\ldots\ldots(1).$$

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* Quaterly Journal, Vol. XVIII, p 73.

† Mathematische Annalen, Bd. XII, pp. 98-122.

‡ Salmon—H. P. Curves, 3rd Ed., p. 254.

§ Clebsch—Binäre Formen, p. 201.

where the V's, linear in f_1, f_2 and f_3, can be found from

$$V_{ik} = \frac{1}{2R} \begin{vmatrix} D_{1i} & D_{2i} & D_{3i} \\ D_{1k} & D_{2k} & D_{3k} \\ f_1 & f_2 & f_3 \end{vmatrix} \text{(where } V_{ik} = -V_{ki}\text{)}.$$

where, $$2R^2 = \begin{vmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{vmatrix} \text{(where } D_{ik} = D_{ki}\text{)}.$$

and $2R^2 . D_{12} = d_{21}\, d_{33} - d_{23}\, d_{31}$ (where $d_{ik} = d_{ki}$).

If the six points of inflexion lie on a conic, it is natural that the researcher will endeavour to develope the cubic (1) into a *rational* curve of the fourth order, *co-trinodal* with the given quartic. The beauty of Brill's memoir is that he actually accomplishes this by means of certain *identities*, arising out of the Theory of Quadratic Forms, demonstrating, once again, the fascinating character of this theory.

Brill, by means of these identities, arrives at the following form of the cubic, which may be written as $\varpi(y_1, y_2, y_3) = 0$.

$$\begin{aligned} i.e.\ \varpi = {} & d_{11}\, y_1^2 (y_1\, D_{23} - y_2\, D_{31} - y_3\, D_{12}) \\ & + d_{22}\, y_2^2 (-y_1\, D_{23} + y_2\, D_{31} - y_3\, D_{12}) \\ & + d_{33}\, y_3^2 (-y_1\, D_{23} - y_2\, D_{31} + y_3\, D_{12}) \\ & + y_1\, y_2\, y_3 (d_{11}\, D_{11} + d_{22}\, D_{22} + d_{33}\, D_{33}) \\ & = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2) \end{aligned}$$

a result of remarkable symmetry.

If Y is the *co-trinodal* to be discovered for the end in view, then it will be possible to establish the identity

$$Y \equiv \varpi . Q + S . P \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

where $\bar{\omega}$ is the *cubic,* Q a *line,* P a *conic* and S the *given* conic. Brill argues that "the co-efficients of the terms $y_1{}^4, y_1{}^3y_2, \ldots\ldots$ in the right-hand side of (3) should collectively vanish. But, for the fulfilment of these nine equations, we have eight magnitudes, *viz*, the ratios of the co-efficients in Q and P, at our disposal. Therefore, *one* of these equations must be a consequence of the rest. If we eliminate the five co-efficients of the conic P, there will remain three equations for the co-efficients of the line Q. Now, as a matter of fact, these three equations can be shown to be consistent with one another"

He thus arrives at

$$Q \equiv d_{11}\, d_{23}\, y_1 + d_{22}\, d_{31}\, y_2 + d_{33}\, d_{12}\, y_3$$

and

$$\begin{aligned} P \equiv & -D_{23}\, d_{23}\, d_{11}\, y_1{}^2 - D_{31}\, d_{31}\, d_{22}\, y^{\ 2} \\ & -D_{12}\, d_{12}\, d_{33}\, y_3{}^2 \\ & +2d_{23}\, (d_{12}\, D_{12} + d_{13}\, D_{13})\, y_2\, y_3 \\ & +2d_{31}\, (d_{23}\, D_{23} + d_{21}\, D_{21})\, y_1\, y_3 \\ & +2d_{12}\, (d_{31}\, D_{31} + d_{32}\, D_{32})\, y_1\, y_2 \end{aligned}$$

and finds that $Y_{11} = -4R^2.\ D_{11}\ (d_{12}D_1\ \ + d_{13}\, D_{13}) \ldots$

$$Y_{23} = R^2(2d_1\ \ d_{13} + 3d_{11}\, d_{23} + 4d_{23}\, D_{23}{}^2)\ ;\ \text{etc.}$$

where $Y_{11}, Y_{12}, Y_{13}, \ldots\ldots$ are the coefficients in Y.

Brill, then, imposes on the members of the *Identity* (3), the quadratic transformation

$$y_1 : y_2 : y_3 := x_2\, x_3 : x_3 x_1 : x_1\, x_2$$

and arrives, obviously, at

$$W_6\, Q_2 + S_4\, P_4 \equiv F.\ x_1{}^2 x_2{}^2 x_3{}^2 \ldots\ldots\ldots\ldots (4)$$

where W_6 (a curve of the *sixth* degree) is what the cubic $\bar{\omega}$ becomes ; Q_2 (a conic passing through the three *nodes*) is what the line Q becomes ; S_4 is the *given quartic* ; P_4 (a curve of the fourth order) is what the conic P

becomes; and F (a conic) is what

$$Y \equiv Y_{11}\, y_2^{\,2} y_3^{\,2} + Y_{22}\, y_1^{\,2} y_3^{\,2} + Y_{33} y_1^{\,2}\, y_2^{\,2}$$
$$+ 2 y_1 y_2 y_3 (Y_{23} y_1 + Y_{13} y_2 + Y_{12} y_3) = 0 \text{ becomes.}$$

$F = 0$ is then obviously the *conic* passing through the six points of inflexion.

The conic does not pass through the three nodes. It does so, when the three nodes are three biflecnodes, and $d_{12} = d_{23} = d_{31} = 0$.

The cubic (1) which is the basis of Brill's Memoir can be found, of course, without the employment of the theory of quadratic forms. We can either consider the points of inflexion on the quartic as points whose polar conic breaks up into two right lines, one of which is the tangent at the inflexion, *or*, we may consider the points where a conic, *circumscribing* the triangle formed by the three nodes, has contact of the *second order* with the conic arrived at by inversion. The second method will give us the cubic.

Writing the circum-conic as

$$S' \equiv l y_2 y_3 + m y_1 y_3 + n y_1 y_2,$$

we have, for the osculation with the given conic S, (*i*) $kS' + S$ breaking up into two right lines, *viz.*, the tangent to S and the chord of intersection through the point of osculation (y_1, y_2, y_3),
and (*ii*) $\Delta' k^3 + \Theta k^2 + \Theta' k + \Delta$ as a perfect cube.

We easily obtain—

$$kl = \frac{2y_1}{y_2 y_3} . V_1$$

$$km = \frac{2y_2}{y_1 y_3} . V_2$$

$$kn = \frac{2y_3}{y_1 y_2} . V_3$$

$$\text{where } V_1 = d_{11}\, y_1 + d_{12}\, y_2 + d_{13}\, y_3$$
$$V_2 = d_{21}\, y_1 + d_{22}\, y_2 + d_{23}\, y_3$$
$$V_3 = d_{31}\, y_1 + d_{32}\, y_2 + d_{33}\, y_3$$
$$k^3 \Delta' + \Delta = 0 \text{ and } 4\Delta' = lmn.$$

These at once lead to $2V_1 V_2 V_3 + \Delta y_1\, y_2\, y_3 = 0$ ... (A)
as equivalent to Brill's cubic, where

$$\Delta \equiv \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix}$$

(y_1, y_2, y_3) being a point on a conic.

We may put $y_1 = a_1\lambda^2 + 2b_1\lambda\mu + c_1\mu^2$

$$y_2 = a_2\lambda^2 + 2b_2\lambda\mu + c_2\mu^2$$

$$y_3 = a_3\lambda^2 + 2b_3\lambda\mu + c_3\mu^2$$

Then, the equation of *any* conic can be written as*

$$\begin{vmatrix} D_{11} & D_{12} & D_{13} & y_1 \\ D_{21} & D_{22} & D_{23} & y_2 \\ D_{31} & D_{32} & D_{33} & y_3 \\ y_1 & y_2 & y_3 & 0 \end{vmatrix} = 0 \ldots (B)$$

where $2D_{ik} = a_i c_k + a_k c_i - 2b_i b_k$

$(i, k = 1, 2, 3)$.

The conic $S \equiv d_{11}y_1^2 + d_{22}y_2^2 + \quad \ldots\ldots \quad = 0$
will be the conic given by (B), if the determinant Δ

$$\text{or } \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix}$$

* Burnside and Panton—Theory of Equations Vol. II, p. 137.

is composed of the inverse constituents of the determinant

$$\Delta' \equiv \begin{vmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{vmatrix}$$

multiplied by a constant factor.

Then we have $\Delta = \Delta'^2$,

$\Delta' D_{11} = d_{22}d_{33} - d_{23}^2$, and similar relations;

$d_{11} = D_{22}D_{33} - D_{23}^2$, and similar relations.

The following identities are derived from the Theory of Invariants and Covariants for a system of three quadratics*

$$\left.\begin{aligned} R.f_1 &= D_{11}V_{23} + D_{21}V_{31} + D_{31}V_{12} \\ R.f_2 &= D_{12}V_{23} + D_{22}V_{31} + D_{32}V_{12} \\ R.f_3 &= D_{13}V_{23} + D_{23}V_{31} + D_{33}V_{12} \end{aligned}\right\} \quad \ldots \quad (a)$$

$$0 = f_1V_{23} + f_2V_{31} + f_3V_{12} \quad \ldots \quad \ldots \quad (b)$$

$$2V_{12}V_{31} = f_1^2D_{23} + f_2f_3D_{11} - f_1f_2D_{31} - f_1f_3D_{12} \quad (c)$$

$$2V_{23}^2 = -f_2^2D_{33} - f_3\ D_{22} + 2f_2f_3D_{23} \quad \ldots \quad (d)$$

where $R = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Also, obviously, $V_1 = 2R.V_{23}$, $V_2 = 2R.V_{31}$, $V_3 = 2R.V_{12}$.

V_{12}, V_{23} and V_{31} are the Jacobians of the system of three Quadratics taken in pairs; D_{11}, D_{22} and D_{33} are the three *invariants* of the Quadratics taken *singly* and D_{12}, D_{13} and D_{23} are the three invariants of the system taken *in pairs*.

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* *Vide*—Salmon's Lessons on Higher Algebra, Third Edition, pp. 171-173.

We have obtained the cubic $\Delta y_1 y_2 y_3 + 2V_1 V_2 V_3 = 0$ with regard to the *adjoint conic* $S = d_{11} y_1{}^2 + \ldots\ldots = 0$ from the condition that $S + kS'$ breaks up into two right lines, where S' is the conic circumscribing the nodal triangle. But, if we put

$S - \lambda(py_1 + qy_2 + ry_3)(y_1 V_1 + y_2 V_2 + y_3 V_3)$ as identical with S', in which the co-efficients of $y_1{}^2$, $y_2{}^2$, $y_3{}^2$ are each nil, we get, considering that $pf_1 + qf_2 + rf_3 = 0$,

$$\frac{d_{11} f_1}{V_1} + \frac{d_{22} f_2}{V_2} + \frac{d_{33} f_3}{V_3} = 0$$

$$\text{or, } \frac{d_{11} f_1}{V_{23}} + \frac{d_{22} f_2}{V_{31}} + \frac{d_{33} f_3}{V_{12}} = 0 \qquad \ldots \quad (5)$$

in terms of the Jacobians. This cubic is easily seen to be equivalent to the cubic arrived at from the other consideration; for, if we make use of the identity (d) above, we get, at once, by substitution of the values of the pairs of products of the Jacobians, the equation of the cubic in Brill's reduced form in terms of the invariants, viz.,

$$d_{11} f_1{}^2 (f_1 D_{23} - f_2 D_{31} - f_3 D_{12}) + \ldots\ldots$$
$$+ f_1 f_2 f_3 (d_{11} D_{11} + d_{22} D_{22} + d_{33} D_{33}) = 0.$$

It will be noted that the use of (5) simplifies Brill's treatment.

It may be interesting to place, side by side, the following correlative properties of the Trinodal quartic :—

(*i*)	The six inflexional tangents touch a conic.	The six points of inflexion lie on a conic (A).
(*ii*)	The six tangents *at* the three nodes touch a conic.	The six points in which the nodal tangents intersect the quartic lie on a conic (B).

(*iii*) The six tangents to the quartic drawn *from* the three nodes touch a conic. | The six points of contact of the tangents drawn *from* the three nodes lie on a conic (C).

We know, moreover, that the eight points of contact of the four bitangents to the trinodal quartic lie on a conic.

The conics (A), (B) and (C) intersect the quartic in two other points, which, with the three nodes determine the conic $d_{11}d_{23}\ x_2x_3+d_{22}d_{31}\ x_1x_3+d_{33}d_{12}\ x_1x_2=0$, which is the conic Q_2 given in Brill's transformation.

As remarked by Salmon, in his Higher Plane Curves, properties of the Trinodal Quartic can be arrived at by means of the Principle of Inversion explained in his book, any line passing through a node being transformed into an inverse line through the same node, any line not passing through a node being transformed into a conic circumscribing the nodal triangle and points on the sides of that triangle being transformed into elements of direction through the nodes. The place of the adjoint conic to which the quartic itself is transformed is important in this Theory, and it is clear that the bitangents to the quartic are transformed into conics passing through the nodes, having double contact with the adjoint conic, and inflexional tangents into conics passing through the nodes, having contact of the second order with the same. The memoir of T. A. Hirst* on the Quadric Inversion of plane curves gives a purely geometrical treatment of the subject in a masterly way.

I have not had access to Meyer's *Apolarität und rationale Curven* which should be consulted to find general methods of treatment of rational curves like the trinodal

* *Vide*—Phil. Magazine, Vol. XIV.

quartic. Of the English works on the subject, Grace and Young's Algebra of Invariants (Arts 249-252) and Glenn's recent book on the Theory of Invariants, Chapter VII may also be usefully consulted, on the application of the Theories of Apolarity and of Combinants to the treatment of rational curves, involving Meyer's Translation Principle by which invariants of binary forms can be translated into co-variants of ternary forms.

The results obtained by Brill are so interesting that a *geometrical* treatment of the problem may be attempted. One way of doing this is by the combination of the methods of Projection and *cyclic* inversion. Project two of the nodes of the Trinodal to the circular points at infinity. The system of conics circumscribing the nodal triangle is transformed into a system of *circles* passing through a *fixed* point, *viz.*, the projection of the third node. We have thus *six* circles passing through a fixed point and *osculating* the conic which is the projection of the adjoint conic. J.C. Malet's theorem*, viz., "the centres of the six circles, which can be described through any point to osculate a given conic, *lie on a conic*" can be easily established by the Invariant theory of conics. The next step is to invert *cyclically* with regard to the fixed point. Taking one of the osculating circles, the end of the diameter through the fixed point lies on a conic (similar to the conic on which the centre lies). We shall call this conic C. It is easy to see that the first negative pedal of the inverse of C is the inverse of the point of osculation. But it is well known that the first negative pedal is the polar reciprocal of the inverse. Therefore, the polar reciprocal of the inverse of the inverse of C is the inverse of the point of osculation. Therefore, the point of osculation

* Vide Casey's Analytical Geometry, 1893, page 471.

lies on the inverse of the polar reciprocal of C. The polar reciprocal of C is a *conic*. Therefore, its inverse is a trinodal quartic having the fixed point as one of the nodes and the circular points at infinity as the other two nodes. Therefore, the point of osculation lies on such a Trinodal, or, in other words, such a Trinodal passes through all the six points of osculation. Now, going back to the original system, this Trinodal corresponds to a quartic *co-trinodal* with the original quartic and the problem is solved. Although, thus, *Geometry* scores a point over *Analysis*, in regard to *simplicity* of solution, it must be admitted that Analysis scores several points in regard to *generality* of treatment and, in particular, in establishing the *actual expression* for the Trinodol quartic which passes through the six points of osculation on the adjoint conic.

I cannot conclude this note better than by inviting students of Geometry to study Quadric transformation in its most general form in Plücker's elaborate work on the *Theorie der Algebraischen Curven*, and *Geometrische Gestalten* by Steiner, Geometer of immortal fame, who replaced *Analysis* by marvellous *Geometry*, just as Lagrange replaced *Geometry* by analysis of great power and beauty.

On the analytical side, the student may consult usefully also the "Notes on the Plane Unicursal Quartic" by R. A. Roberts in Volume XVI of the Proceedings of the London Mathematical Society, pp. 44-60.

The following result, among others, is established :—

The tangential equation of the quartic—

$$x^2y^2+y^2z^2+z^2x^2+2xyz\,(ax+by+cz)=0 \text{ is}$$

$$(8\rho^3+27\Delta^2\lambda\mu\nu+9\Delta\rho\Sigma)^2-(4\rho^2+3\Delta\Sigma)^3=0,$$

where $\lambda^2+\mu^2+\nu^2-2\lambda\mu\nu$ ... ... $=\Sigma$

$$(a-bc)\lambda+(b-ca)\mu+(c-ab)\nu=\rho,$$

$$1+2abc-a^2-b^2-c^2=\Delta,$$

where λ, μ, ν are tangential coordinates and a, b, c parameters.

From the tangential equation it is evident that

$$4\rho^2+3\Delta\Sigma=0$$

represents a conic *touching* the six inflexional tangents and Σ is the conic which touches the six tangents at the nodes. These two conics have double contact with each other, the point represented by ρ being the pole of the chord of contact.

There are other interesting properties to this Note, but the problem of the "six-inflexional-points-conic" is not dealt with.

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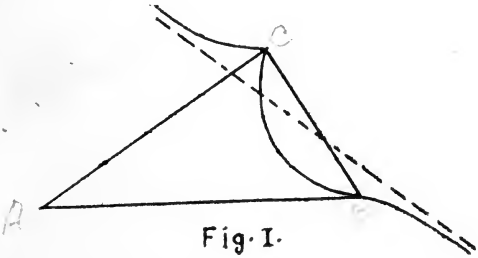
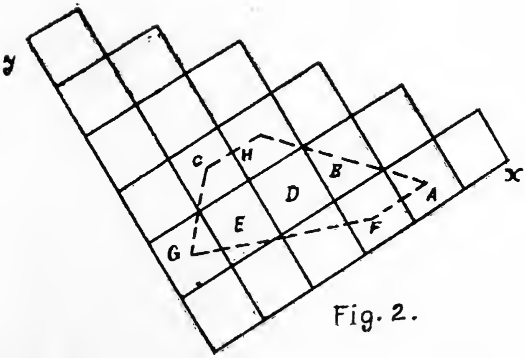


Fig. I.



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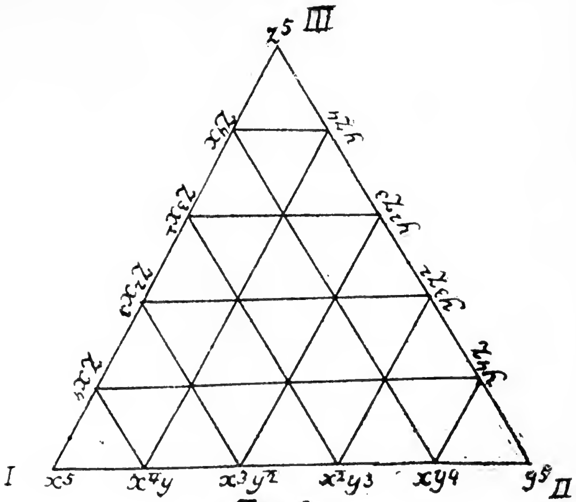

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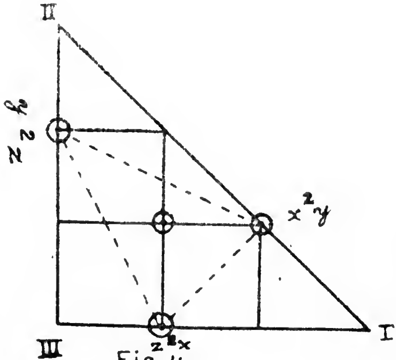

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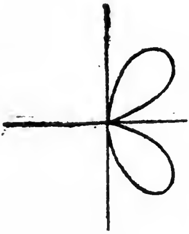

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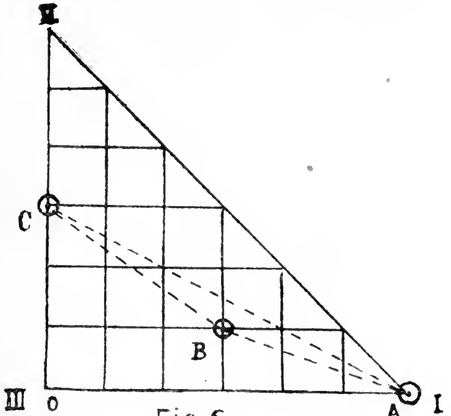

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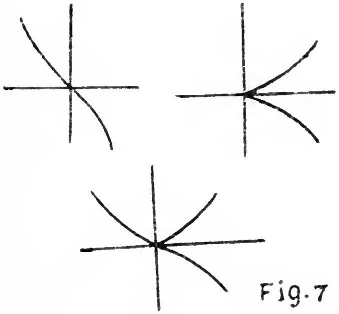

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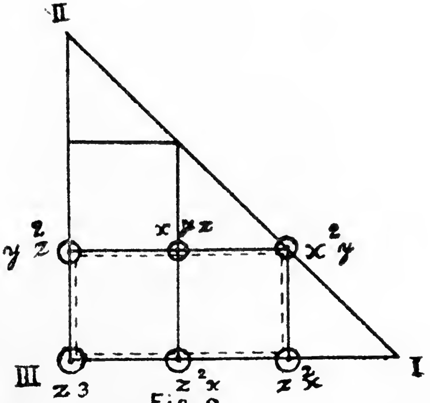

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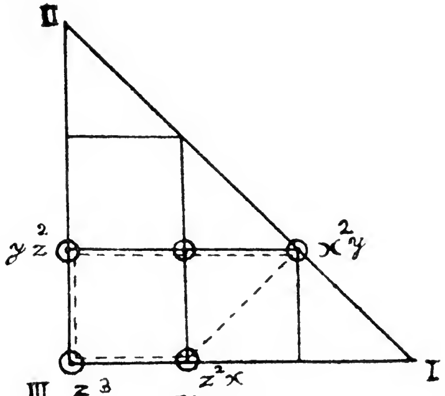

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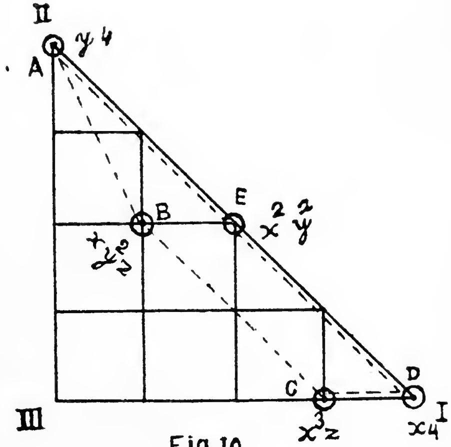

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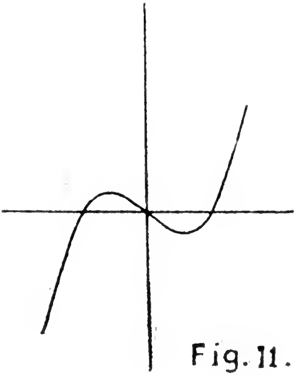

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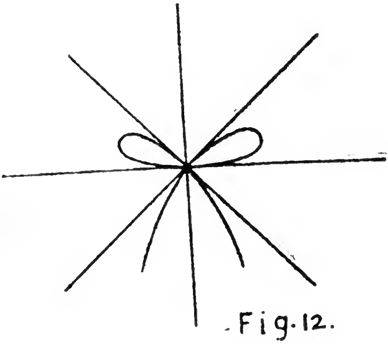

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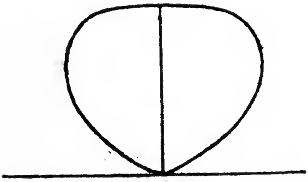

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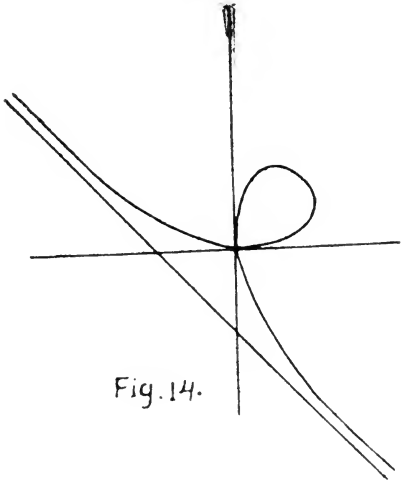

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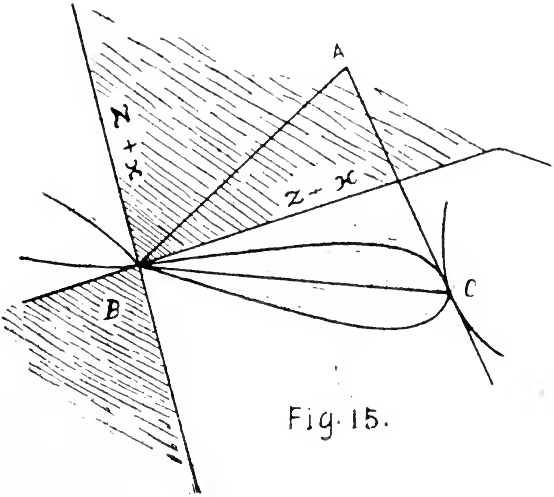

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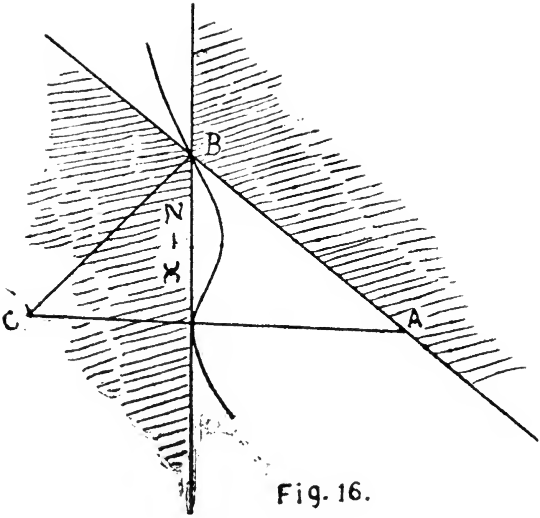

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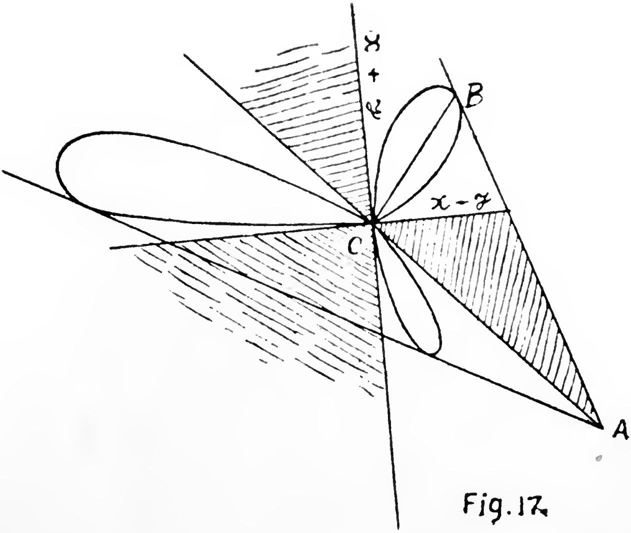

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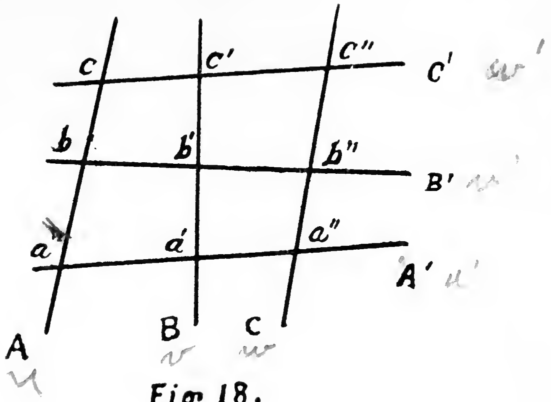

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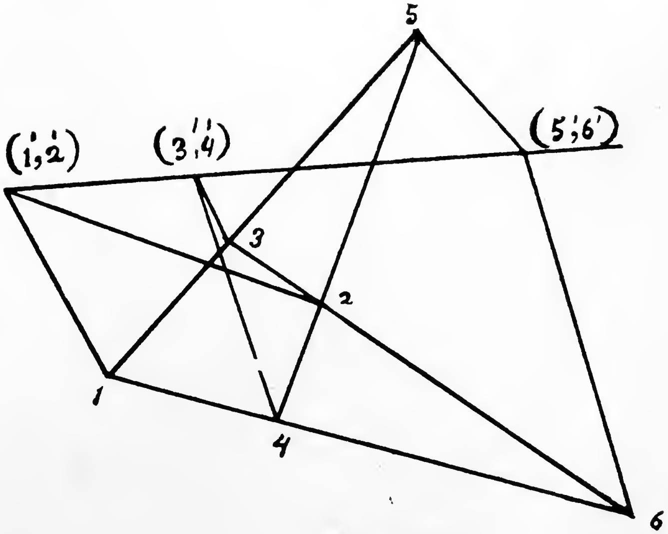

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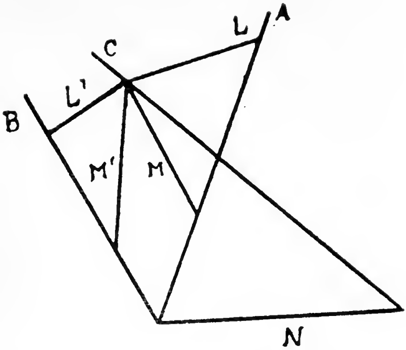

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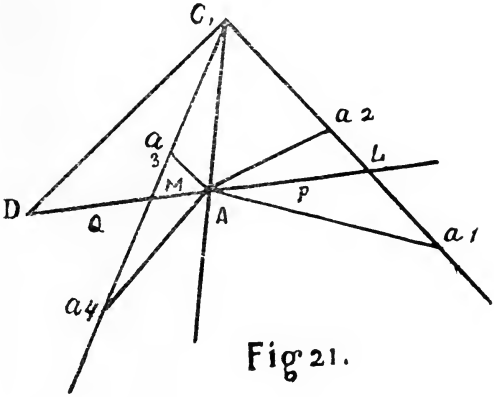

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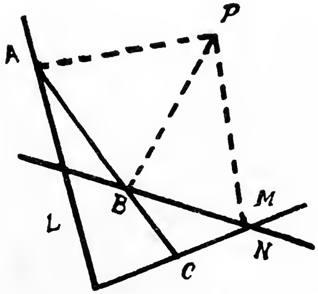

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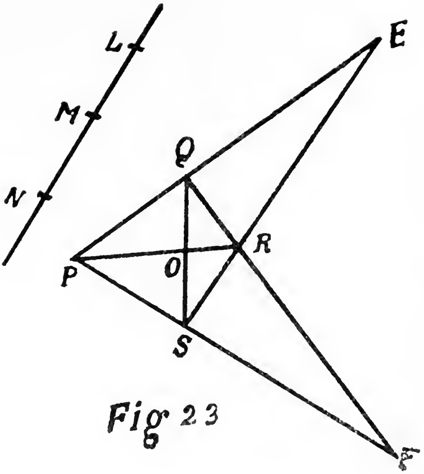

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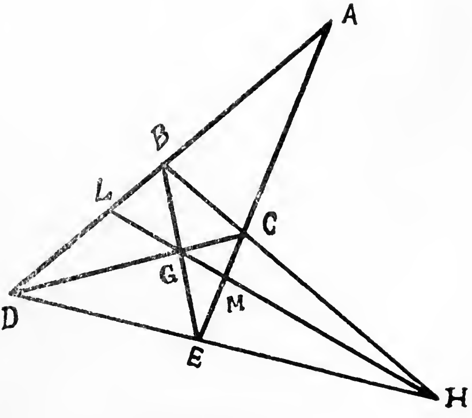

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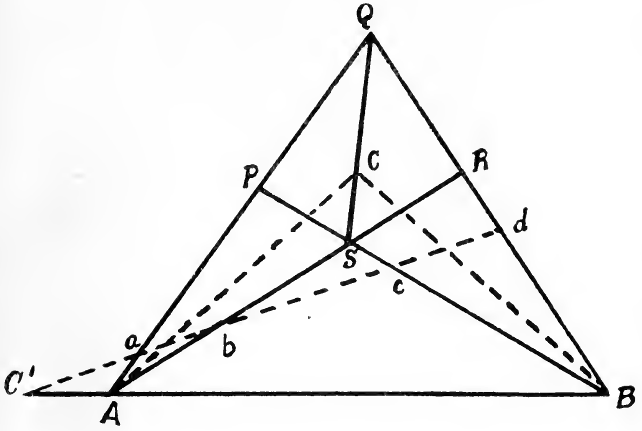

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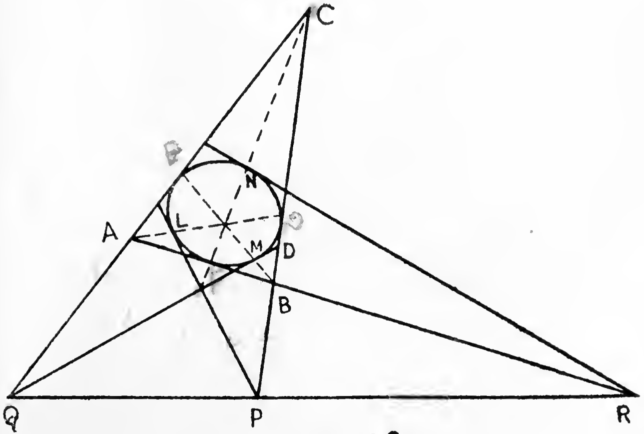

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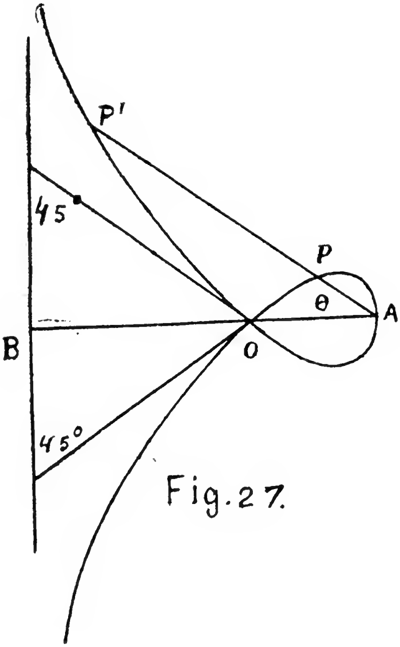

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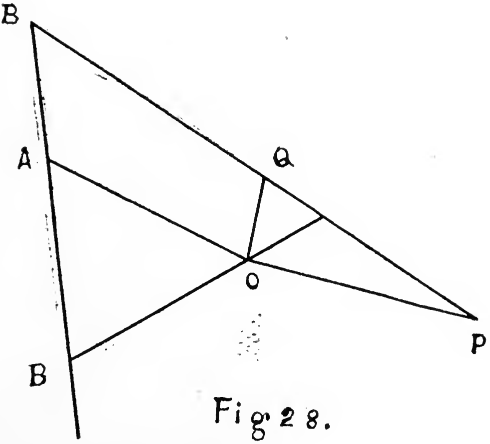

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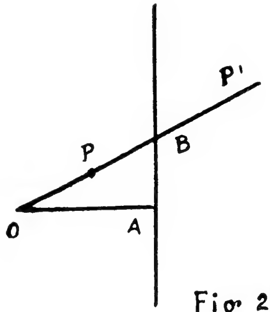

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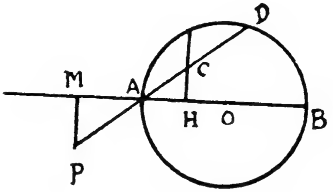

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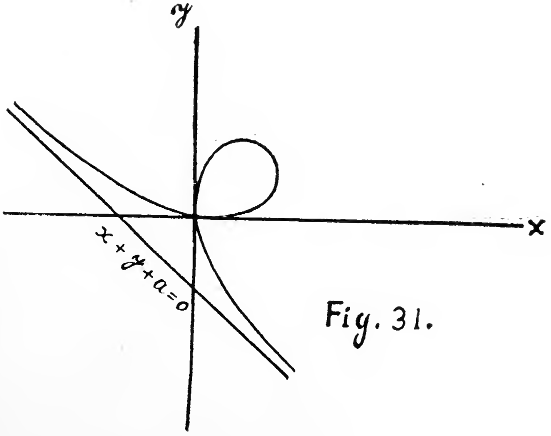

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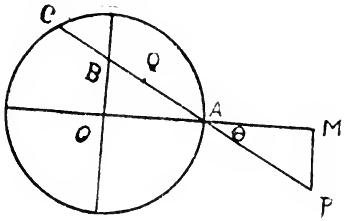

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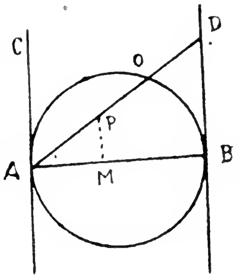

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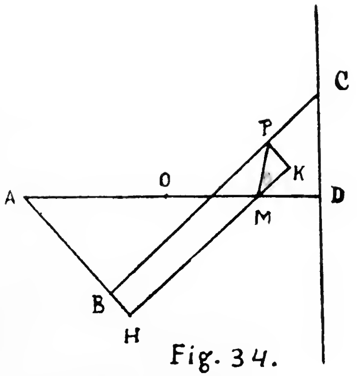

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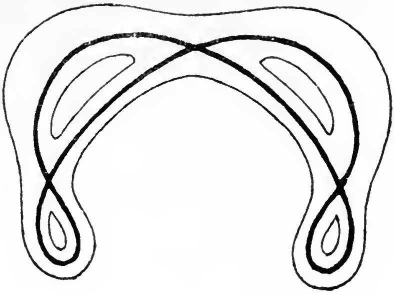

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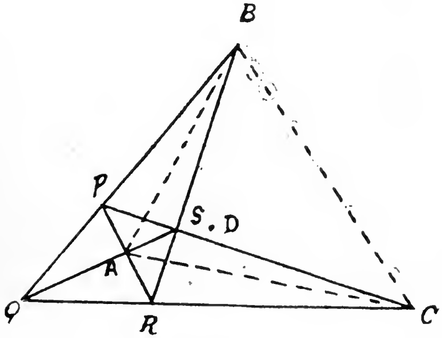

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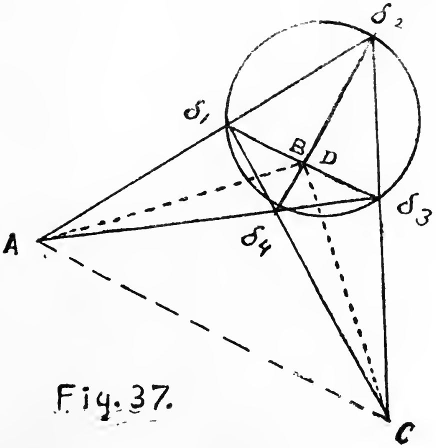

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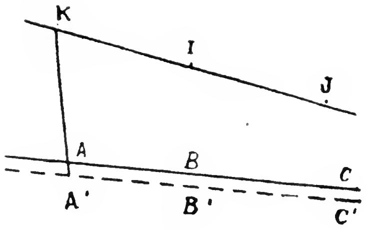

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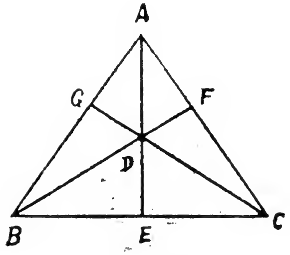

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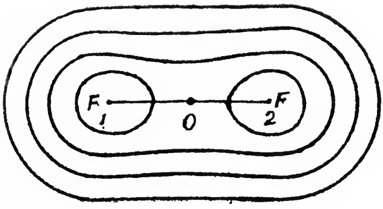

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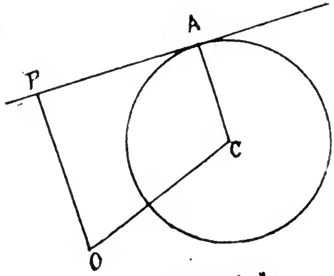

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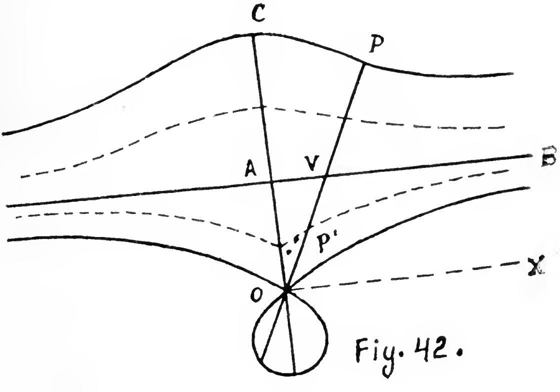

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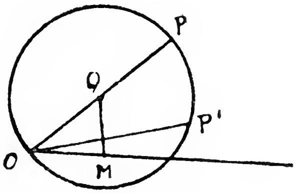

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